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MODEL REDUCTION

AND

REDUCED ORDER CONTROL LAW DESIGN

by

© ROBERT G. WILSON

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

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EDMONTON, ALBERTA

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled MODEL REDUCTION AND REDUCED ORDER CONTROL LAW DESIGN submitted by ROBERT G. WILSON, B. ENG., in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

It is often desirable to simplify multivariable feedback controllers designed using modern control theory so that they will be easier and more practical to apply to the control of industrial processes. Typical simplifications include linearization, assumption of time-invariant models, and reduction of the order of the governing equations. This thesis is concerned primarily with:

1. The simplification of linear time-invariant state space models using modal analysis techniques to eliminate selected state variables followed by the design of multivariable controllers based on these reduced models.
2. The use of modal analysis techniques to eliminate selected state variables from a "high order" control law designed using the original process model.
3. The use of least squares techniques, instead of the modal analysis mentioned above, to fit the low order system (model or control law) to data generated by exciting the high order system with a sequence of random numbers.
4. The development of computer programs to implement the above techniques plus examination of related theoretical points such as the equivalence of different reduction techniques.

All of the techniques developed in this work were applied to the computer controlled pilot plant evaporator at the University of Alberta that was used in previous multivariable control studies. Simulated and/or experimental data are presented to show the relative

performance of different process models and controllers. Specific contributions and conclusions are:

1. Existing modal approaches to the reduction of continuous-time models are extended for use with discrete-time models. Several of the basic modal results presented in the literature are shown to be equivalent.
2. In calculating a reduced order discrete-time model from a high order continuous-time model using the modal approach, the relative order of the model reduction and the model discretization steps is unimportant, since the same reduced order model results. However, using the least squares approach, and the same sequence of random numbers, different reduced order discrete-time models result when the order of the reduction step and the discretization step is interchanged.
3. Control laws designed by eliminating selected state variables from a high order control law using the modal analysis were the only controllers to give consistently good control of the evaporator.
4. When the least squares approach with random data is used to simplify control laws, the result obtained by minimizing the sum of the squares of the difference between the original control vector and the desired reduced order control vector is the same as that obtained by minimizing the sum of the squares of the difference between the closed-loop responses of the high order model, when controlled by the high order controller and by the desired reduced order controller.

In general, it was concluded that a multivariable control law, such as the tenth order optimal controller designed for the pilot plant evaporator, can be simplified by the elimination of selected state variables and that the resulting control schemes can be easily and successfully implemented on actual process units.

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CHAPTER ONE

INTRODUCTION

1.1 GENERAL

According to the Encyclopedia Britannica, a control system is any means, natural or artificial, by which a variable quantity or set of variable quantities is caused to conform, more or less, to some prescribed norm [29]. The most common type of control system presently implemented in the chemical and petroleum industry is a single variable feedback system applied through the use of electronic and pneumatic instrumentation. The theory and application of these single loop controllers has been discussed in detail in many text-books [5,6,11,22]. They are relatively easy to apply to the process and can provide good control of the variable of interest. However, difficulties can develop when process variables interact or when the control variable for one section of the plant is a feed variable for another section. Some modifications, such as feedforward control, help to overcome this problem. A more recent approach to eliminating these interactions is the use of a multivariable control scheme.

Optimal multivariable control and its theoretical potential has been discussed at length [3,13,14,21]. However, its application to the control of a physical process is more difficult than the application of single variable controllers. This is because each control value is calculated as a function of many process values (states). This practical application has become feasible with the development of the control computer capable of performing the required

on-line calculations. Also, in the application of optimal multivariable control, values must be obtained for each of the process variables that are in the state vector of the process model. Usually, these cannot all be measured so that estimated values must be obtained. This, in turn, requires more on-line computer calculation time and so increases the loading on the computer. An alternate approach for dealing with unmeasurable states is to eliminate them from the control law. It is this approach which is treated in this thesis, using two general techniques. In the first, the unmeasurable states are eliminated from the process model and the resulting model is used to calculate the optimal multivariable control law (model reduction); in the second, the unmeasurable states are eliminated from the optimal control law itself (control law reduction).

Previous control studies at the University of Alberta include the following. Andre [2] built a pilot plant evaporator and controlled it with conventional single loop feedback electronic instruments. Wilson [25] controlled the pilot plant with electronic feedforward plus feedback controllers. Jacobson [12] and Fehr [7] performed similar studies using direct digital control on the newly acquired IBM 1800 Data Acquisition and Control Computer [8,28]. They also studied inferential control of the product concentration; controlling the concentration using calculated values instead of measured values. Newell [17] applied optimal multivariable control with proportional feedback, integral feedback, feedforward and setpoint control modes. Additional advanced control techniques have been applied to the evaporator by Nieman [18] (optimal state driving control), Oliver

[19] (model reference adaptive control), Alevisakis [1] (time delay compensation) and Hamilton [10] (Kalman filters and Luenberger observers).

Control studies at the University of Alberta using a pilot plant distillation column [4,16,20,23] and identification studies on a heat exchanger [15,24] have also been done.

1.2 SCOPE AND STRUCTURE OF THE THESIS

This thesis treats the general problem of controlling a process which has unmeasurable states in the process model. The basic model for this study is a tenth order linear model of a pilot plant evaporator which was obtained by a numerical linearization of the nonlinear model developed by Newell [17]. The general form of the model is in one of two forms; the continuous-time form as

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u} + \underline{\underline{D}}\,\underline{d} \tag{1.1}$$

$$\underline{y} = \underline{\underline{C}}\,\underline{x} \tag{1.2}$$

or the discrete-time form as

$$\underline{x}(j+1) = \underline{\underline{\Phi}}\,\underline{x}(j) + \underline{\underline{\Delta}}\,\underline{u}(j) + \underline{\underline{\Theta}}\,\underline{d}(j) \tag{1.3}$$

$$\underline{y}(j) = \underline{\underline{C}}\,\underline{x}(j) \tag{1.4}$$

where $\underline{x}$ is the n-dimensional state vector

$\underline{u}$ is the m-dimensional control vector

$\underline{d}$ is the q-dimensional disturbance vector

$\underline{y}$ is the p-dimensional output vector

$\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$, $\underline{\Phi}$, $\underline{\Delta}$, $\underline{\Theta}$ are constant coefficient matrices of appropriate dimensions

j is a counter such that $\underline{x}(j) = \underline{x}(t)$ when $t = jT$ and

T is the discrete-time interval

The basic control law used is a linear proportional feedback, integral feedback, feedforward plus setpoint controller which results from calculations using dynamic programming techniques to minimize a quadratic performance criterion. Unless otherwise stated, the control laws are discrete-time controllers so that they can be applied to the control of the pilot plant evaporator using a digital computer.

The topics discussed in this thesis can be outlined as part of the general scheme presented in Figure 1.1. The model reduction steps represented by path one for a continuous-time model and by path four for a discrete-time model are treated in Chapters Three, Four, and Six. Procedures for control law reduction represented by path eight are developed in Chaptess Five and Six. The other topic discussed in the thesis is the calculation of a reduced order control law as the optimal controller for a reduced order model which had been obtained by model reduction techniques (Part of Chapter 6). This is shown in Figure 1.1 by path four followed by path seven.

The thesis is structured in the general form of a series of publications dealing with the methods outlined in Figure 1.1. Earlier versions of several chapters have been published, as noted, in the discussion which follows.

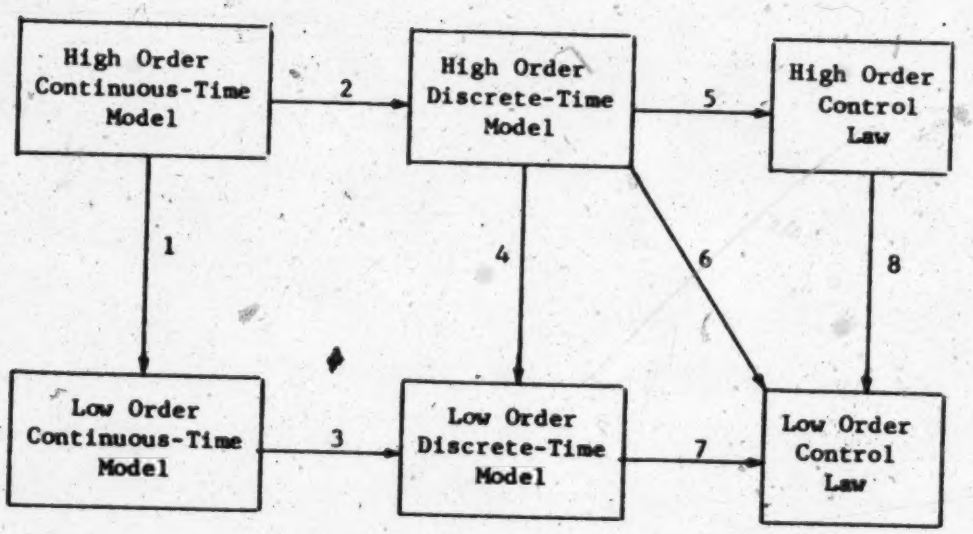


FIGURE 1.1 PATHS FOR THE CALCULATION OF LOW ORDER MODELS AND LOW ORDER CONTROL LAWS FROM HIGH ORDER MODELS

In Chapter Two a GEneral Multipurpose, Simulation and COntrol PackagE (GEMSCOPE) is presented. For ease in simulation and design calculations, a collection of computer programs was assembled into one integrated package. This package allows the control system designer to specify a model, perform various calculations on his model (model reduction, calculate the discrete-time model, calculate a discrete-time model with a different sampling interval), design or specify a control law and calculate a time domain response. An earlier version of this chapter has appeared in the literature [9].

In Chapter Three, several existing approaches to model reduction for continuous-time models are extended for use in reducing the order of discrete-time models. The resulting reduced order discrete-time models calculated using the modal approach from a high order continuous-time model along two paths of Figure 1.1 (Path one followed by path three and path two followed by path four) are compared. An earlier version of this paper appeared in reference [26].

The subject of model reduction using least squares to fit the response of the reduced order model to that of the high order model is discussed in Chapter Four, where an alternate method of obtaining the data used by least squares is presented.

A modal approach to control law reduction (path eight of Figure 1.1) is presented in Chapter Five. An earlier version of this paper appeared in reference [27].

Chapter Six includes several aspects of both model reduction and control law reduction. The modal approach to model reduction is

discussed in more detail and several of the results presented in the literature are compared. Three approaches to reduced order control law design are discussed and applied to the control of a pilot plant evaporator.

Finally, some overall conclusions and recommendations for future work are presented in Chapter Seven.

CHAPTER TWO

DESCRIPTION OF A COMPUTER PACKAGE FOR CONTROL SYSTEM DESIGN

ABSTRACT

A GEneral, Multipurpose Simulation and COntrol PackagE (GEMSCOPE) is described which assists with the design, analysis, and digital simulation of linear, time-invariant dynamic systems. The process model may be defined as transfer functions or in the form of a standard state space model in either the continuous-time form or the discrete-time form. Options are included for generating the optimal control matrices, determining the state feedback matrix for non-interaction, reducing the order of the model and of the control law, and calculating the open or closed-loop system response. GEMSCOPE was used for all the model and control law calculations used in this thesis.

2.1 INTRODUCTION

GEMSCOPE (a GEneral, Multipurpose, Simulation and COntrol PackagE) is an integrated series of computer programs to aid in the simulation of linear models and in the design of control systems for these models. It has been designed to run, either as a batch job or from a time sharing terminal, on an IBM 360 model 67 computer operating under the University of Michigan, time-sharing terminal oriented system (MTS). It has been set up in order to make use of some of the file manipulation programs available with the MTS system. Some of the programs in GEMSCOPE were used in recent studies by Newell [20,21] and most of the programs were used in the design and simulation calculations to be outlined in this thesis.

The basic data used in GEMSCOPE as well as some of the basic programs (data input and output, calculation of a state space model from a transfer function model, calculation of a discrete-time model from a continuous-time model, calculation of a time domain response) are those which made up the Continuous Systems Analysis Program (CSAP) described by Agostinis and Fisher [1,2,3]. GEMSCOPE itself is described in a series of research reports. The GEMSCOPE User's Manual [29] describes the use of the existing programs as well as the required data. Other reports summarize the file structure and the subroutines used in GEMSCOPE [30], indicate the required steps to add a new program to GEMSCOPE [31], and outline a sample problem using GEMSCOPE [29].

Many other digital simulation and design programs exist. One of the first was developed by Kalman and Englar [15] and others are described in textbooks by Franks [9] and Melsa [19]. Several companies

and organizations also have their own programs for internal use. This chapter, therefore, describes an example of such programs, not a unique program, by discussing the basic philosphy used to set up GEMSCOPE and briefly outlining the main programs in the package.

2.2 GEMSCOPE PROGRAMS

GEMSCOPE is a series of separate programs which share common disk storage files and can be executed either singly or sequentially to perform operations such as model definition, control calculations or simulation of time domain responses. From the point of view of simulation the important programs are:

1. Specification of the programs to be executed
2. Definition of the state space model
3. Calculation of the state difference equation
4. Calculation of the time domain response.

These programs are shown in the center column of Figure 2.1. However, the usefulness of GEMSCOPE is increased significantly by the functions represented by the blocks in the right most column of Figure 2.1 which includes:

5. Generation of an equivalent state space form for models entered as transfer functions
6. Reduction of the order of the state space model
7. Generation of state feedback required for non-interaction
8. Calculation of the control matrices for optimal regulatory control
9. Simplification of high order control laws

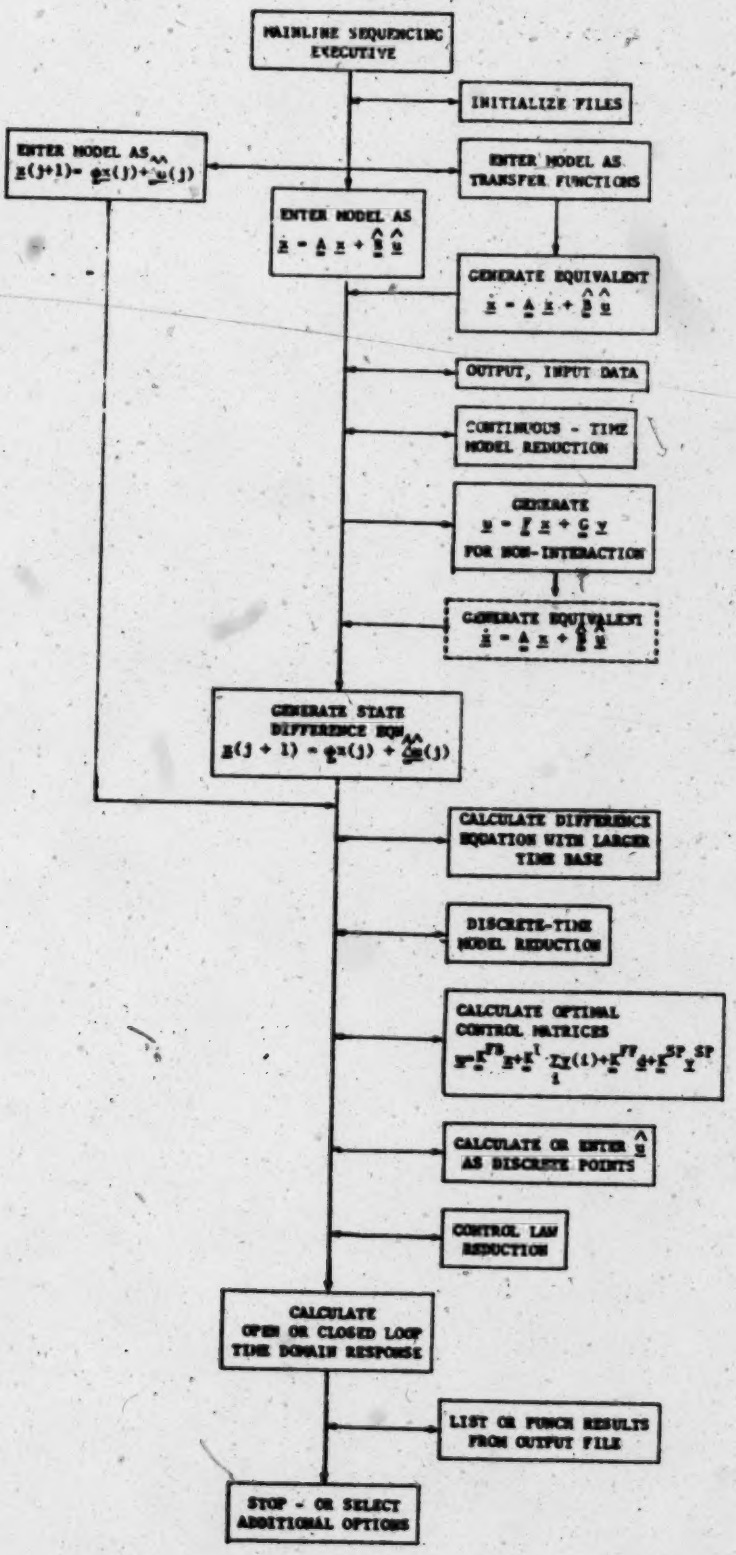

MAINLINE SEQUENCING EXECUTIVE
INITIALIZE FILES
ENTER MODEL AS
ENTER MODEL AS TRANSFER FUNCTIONS
ENTER MODEL AS
GENERATE EQUIVALENT
OUTPUT, INPUT DATA
CONTINUOUS - TIME MODEL REDUCTION
GENERATE
FOR NON-INTERACTION
GENERATE EQUIVALENT
GENERATE STATE DIFFERENCE EQN
CALCULATE DIFFERENCE EQUATION WITH LARGER TIME BASE
DISCRETE-TIME MODEL REDUCTION
CALCULATE OPTIMAL CONTROL MATRICES
CALCULATE OR ENTER AS DISCRETE POINTS
CONTROL LAW REDUCTION
CALCULATE OPEN OR CLOSED LOOP TIME DOMAIN RESPONSE
LIST OR PUNCH RESULTS FROM OUTPUT FILE
STOP - OR SELECT ADDITIONAL OPTIONS

FIGURE 2.1 SIMPLIFIED FLOW DIAGRAM OF GEMSCOPE

10. Utilization of a discrete control vector for open-loop or optimal state-driving.

These methods are discussed in the following sections. Some of the "utility functions" for performing tasks like listing output data are shown in Figure 2.1 but are not discussed further.

The GEMSCOPE programs in their present form consist of about 7700 FORTRAN source cards and in object form occupy about 140,000 words of storage. Core memory requirements vary because the IBM 360/67 uses dynamic memory paging but a typical average would be 30,000 words. Central Processing Unit time requirements vary with each run but average about 40 seconds.

2.2.1 Job Specification

It should be emphasized that each program in GEMSCOPE is a separate mainline program and can be executed individually or in any sequence that the user specifies. The mainline sequencing executive permits the user to specify which program blocks he wants executed by simply entering a series of integer code numbers either via punched cards (if he is running GEMSCOPE as a batch program) or via a keyboard on a time sharing terminal. The executive then generates the appropriate series of job control statements and stores them in a file. Special files of job control statements can be retained for frequently used combinations of programs. The computer monitor system will then execute the GEMSCOPE programs in accordance with the job control statements in this file and then exit, or return to the executive. This

approach maximizes flexibility, facilitates reassignment of input-output devices and files for each program block, and makes it a simple matter (since each block is a self contained mainline program) to add, modify or delete programs. This approach also allowed the full use of the computer system support programs, such as file manipulation routines, which are specified by the use of job control statements.

2.2.2 State Space Model Specification

The system to be simulated can be specified in standard state space form (order ≤ 30) or, as described later, in the classical transfer function and block diagram form. For matrix input the coefficient matrices are entered by the user or obtained from previously established input files in the form of a continuous-time system as,

$$\dot{\underline{x}} = \underline{A}\underline{x} + \hat{\underline{B}}\hat{\underline{u}} \tag{2.1}$$

$$\underline{y} = \underline{C}\underline{x} + \underline{E}^{*}\underline{w} \tag{2.2}$$

or in the form of a discrete-time system as Equation (2.2) and

$$\underline{x}(j+1) = \underline{\Phi}\underline{x}(j) + \hat{\underline{\Delta}}\hat{\underline{u}}(j) \tag{2.3}$$

where: $\underline{x}$ = state vector

$\hat{\underline{u}}$ = input vector

$\underline{y}$ = output vector

$\underline{w}$ = input plus delayed vector

$\underline{A}$, $\hat{\underline{B}}$, $\underline{C}$, $\underline{E}^{*}$, $\underline{\Phi}$, $\hat{\underline{\Delta}}$ = constant coefficient matrices of appropriate dimension

j is a counter for time intervals such that $\underline{x}(j)$ denotes $\underline{x}(t)$ at $t = jT$, where T is the discrete time interval.

2.2.3 State Difference Equation

When a state space model in the continuous-time form of Equation (2.1) is available, the equivalent discrete-time model of Equation (2.3) may be needed for some applications. Thus, a program is available to calculate Equation (2.3) from Equation (2.1).

Equation (2.3) agrees exactly with Equation (2.1) at each sampling point if the input vector, $\hat{\underline{u}}$, is constant over the sampling interval (which is true when $\hat{\underline{u}}$ originates from a digital control computer), otherwise an approximation is involved. This discrete form is particularly convenient for digital calculation and $\underline{\underline{\Phi}}$ and $\hat{\underline{\underline{\Delta}}}$ are available for use by other programs.

The fundamental matrix, $\underline{\underline{\Phi}}$, can be evaluated using either a power series expansion or the Cayley-Hamilton technique as described by Agostinis [1] or in textbooks such as Ogata [25] and Saucedo and Schiring [28]. Implementation of the latter technique includes algorithms for determining real, complex and/or repeated eigenvalues of non-symmetric $\underline{\underline{A}}$ matrices. The $\hat{\underline{\underline{\Delta}}}$ matrix is obtained by numeric integration of $\underline{\underline{\Phi}}\hat{\underline{\underline{B}}}$ over the interval T.

This program to calculate the discrete-time model was originally part of CSAP [1,2,3].

The specification of the time interval, T, has a pronounced effect on the numerical stability of some of the algorithms, is usually the dominant factor affecting simulation time and determines the resolution or precision of the time domain response. Therefore, to provide

additional flexibility, a program has been included to modify the previously calculated coefficient matrices in Equation (2.3) so that they apply to an interval mT (where m is an integer) instead of T and

$$\underline{\Phi}(mT) = \underline{\Phi}^{m}(T) \tag{2.4}$$

$$\hat{\underline{\Delta}}(mT) = \left[\underline{\Phi}^{m-1}(T) + \ldots + \underline{\Phi}(T) + \underline{I}\right]\hat{\underline{\Delta}}(T) \tag{2.5}$$

In general, since use of Equations (2.4) and (2.5) is considerably faster than recalculating $\underline{\Phi}$ and $\hat{\underline{\Delta}}$, it is most expedient to specify a small interval T for the first evaluation of $\underline{\Phi}$ and $\hat{\underline{\Delta}}$. Relatively large multiples, mT, can be used for exploratory studies and then m can be reduced until satisfactory results are obtained.

2.2.4 Time Domain Response

The time domain response is evaluated directly using Equations (2.2) and (2.3). The initial state vector, if not supplied by the user, is initialized to zero. The elements of the input vector, $\hat{\underline{u}}$, can be defined by selecting standard options as step, ramp, sinusoid, or pulse functions, or defined by a discrete-time series of point values. The control variables can also be specified as originating from a control law as discussed in the section on optimal multivariable control (Section 2.2.8). For each time domain response, this program also calculates the numerical value of the quadratic performance index, J, which is optimized in the calculation of the optimal multivariable control matrices. This performance index is also discussed in Section 2.2.8.

The basic form of this program was part of CSAP [1,2,3] and has been revised slightly for use in GEMSCOPE.

2.2.5 Transfer Function Model

Most of the current research work in areas such as optimal control is performed using state space techniques. However, a great deal of practical design and analysis of control systems is still done using classical techniques based on transfer functions and block diagrams. This approach has been the subject of many textbooks such as that by Coughanowr and Koppel [5]. Therefore, to serve as a convenient link between the two approaches, a program was written which would accept a problem defined in terms of transfer functions and a coded equivalent of the user's block diagram, and produce an equivalent state space representation in the form of Equations (2.1) and (2.2). There are no arbitrary restrictions on the form of the block diagram except the dimensions specified for vectors and matrices in the digital program. Pure time delays, isolated differentiation terms and isolated integral terms can be handled. This program also can specify control constants for standard single loop control laws using the transfer functions, the block diagram and several different criteria such as the Ziegler-Nichols frequency response technique, the Cohen and Coon response curve technique and others. The program then rearranges these control constants into a matrix form so that it can be used in the same form as the optimal multivariable control law discussed in Section 2.2.8.

This program was originally part of CSAP [1,2,3] and was added to GEMSCOPE by Park [27].

2.2.6 Model Reduction

Model reduction is frequently desirable to reduce the computational requirements when a large series of simulations is to be done, or to eliminate states which are not of interest and/or are not physically measurable. Also, when the model, or control strategies based on it, are to be implemented on a computer controlled process, then model reduction will reduce the amount of real-time calculations. Therefore, two programs have been included in GEMSCOPE to reduce the order of a state space model by calculating a reduced order model whose state vector is a subset of the vector $\underline{x}$. Each of these programs permit the user to reduce the order of either the continuous-time model of Equation (2.1) or the discrete-time model of Equation (2.3).

In the first program, model reduction is accomplished using a modal analysis of the high order model by eliminating the least significant eigenvalues. The user has the option of specifying the continuous-time reduction approaches of Marshall [18], Davison [6], Fossard [8] and Graham [12] as well as the equivalent discrete-time extensions of these approaches which are discussed in Chapter Three. It is also shown in Chapter Three, that using these modal approaches, the continuous-time model reduction followed by conversion to the discrete-time form is equivalent to discretization of the high order model followed by discrete-time model reduction.. Model reduction using the modal analysis is discussed in detail in Chapters Three and Six.

In the second program, a least squares analysis, and a sequence of uniformly distributed random numbers are used to calculate the reduced order model. Least squares is used to "fit" a reduced order model to

the high order model using data which are calculated from the high order model and the random numbers. This approach is discussed in detail in Chapter Four.

2.2.7 Non-interaction

For the control of multivariable processes it is frequently not the dynamic relationship between a given pair of input and output variables that causes difficulty but rather the interaction between these variables and other state variables. One design approach is to develop an augmented plant model such that there is no interaction between the new reference variables and the output variables.

The decoupling program included in GEMSCOPE is a modified version of one developed by Gilbert and Pivnichny [11]. The program first tests to see whether the model meets the necessary and sufficient conditions for non-interaction by state feedback as described by Falb and Wolovich [7]. If the model, given in the form of Equations (2.1) and (2.2) can be made non-interacting the program synthesizes a control law in the following form:

$$\underline{u} = \underline{\underline{F}}\underline{x} + \underline{\underline{N}}\underline{v} \tag{2.6}$$

The reference input vector $\underline{v}$ is related to the original process outputs by the following equations written in terms of Laplace transforms

$$\underline{y}(s) = \underline{\underline{W}}(s)\ \underline{v}(s) \tag{2.7}$$

The transfer matrix $\underline{\underline{W}}(s)$ is diagonal and the elements are specified by the user, as functions of two vectors, so that the system defined by Equation (2.7) has the desired dynamic response. This

fixes the elements of $\underline{\underline{F}}$ and $\underline{\underline{N}}$. The basic theory behind the algorithm has been described by Gilbert [10]. The user then specifies the augmented model in the state space form of Equation (2.1) and (2.2) and re-enters GEMSCOPE for further simulation and/or control.

This approach to decoupling control has been applied to a fifth order distillation column model by Newell [20,21].

2.2.8 Optimal Multivariable Controllers

Optimal multivariable controllers with proportional-plus-integral feedback, feedforward compensation for constant disturbances, $\underline{d}$, and provision for driving the process from its current position to setpoint values, $\underline{y}^{SP}$, can be implemented using a control law of the form

$$\underline{u}(j) = \underline{\underline{K}}^{FB}\underline{x}(j) + \underline{\underline{K}}^{I}\sum_{i=0}^{j-1}\underline{y}(i) + \underline{\underline{K}}^{FF}\underline{d}(j) + \underline{\underline{K}}^{SP}\underline{y}^{SP} \qquad (2.8)$$

where $\underline{u}$ and $\underline{d}$ are defined as partitions of vector $\hat{\underline{u}}$ as

$$\hat{\underline{u}} = \begin{bmatrix} \underline{u} \\ \underline{d} \end{bmatrix} \qquad (2.9)$$

and $\underline{\underline{K}}^{FB}$, $\underline{\underline{K}}^{I}$, $\underline{\underline{K}}^{FF}$, $\underline{\underline{K}}^{SP}$ are the controller matrices. For the special case of diagonal control matrices, Equation (2.8) represents a set of conventional single variable control laws.

A program is included in GEMSCOPE to allow the user to supply these controller matrices or to calculate them for the partitioned form of Equation (2.3) as

$$\underline{x}(j+1) = \underline{\phi}\,\underline{x}(j) + \underline{\Delta}\,\underline{u}(j) + \underline{\Theta}\,\underline{d}(j) \qquad (2.10)$$

where $\hat{\underline{\Delta}} = (\underline{\Delta}, \underline{\Theta})$. This calculation uses recursive relations developed by applying the concepts of discrete dynamic programming to the model of Equation (2.10) and a quadratic performance criterion of the following form:

$$J = \beta^N (\underline{x}(N) - \underline{x}^{SP})^T \underline{S}\, (\underline{x}(N) - \underline{x}^{SP}) + \sum_{i=1}^{N} \beta^i \left\{ (\underline{x}(i) - \underline{x}^{SP})^T \underline{Q}\, (\underline{x}(i) - \underline{x}^{SP}) + \underline{u}^T(i-1)\underline{R}\,\underline{u}(i-1) \right\} \qquad (2.11)$$

where β, a time weighting factor, and $\underline{Q}$, $\underline{R}$, and $\underline{S}$, state or control weighting matrices, are specified by the user. The dynamic programming solution calculates the proportional feedback, feedforward, and setpoint controller matrices directly from the model in Equation (2.10), and evaluates the integral matrix, from an augmented state space model where the original state vector, $\underline{x}$ is augmented by $\underline{x}^I$ where $\underline{x}^I$ is defined by

$$\underline{x}^I(j) = T \sum_{i=0}^{j} \underline{y}(i) \qquad (2.12)$$

and where $\underline{y}(i)$ is defined by Equation (2.2) with $\underline{E}^* = \underline{0}$. Newell et al, [20,22,23] showed that this method of generating control matrices is relatively straightforward, practical, easy to implement and has given outstanding results in both simulated and experimental control studies. Optimal control laws, calculated by this method, are used in Chapters Five and Six.

2.2.9 Control Law Reduction

When optimal control theory is used to obtain a multivariable

control law for a linear state space model in the form of Equation (2.1) or (2.3), the resulting control law is a function of the entire state vector. This is shown by the control law in Equation (2.8). In many cases, the state vector contains so many elements that the resulting control law is too expensive to apply to the control of the actual process using a control computer. This expense is due to the large on-line computer time required to measure all the elements of this large state vector, or to calculate any which cannot be measured.

In order to reduce the complexity of the control law, GEMSCOPE contains two programs to calculate low order control laws which are a function of a subset of the original state vector. Each of these programs make use of the high order control law. The resulting reduced order control law, being a function of fewer state variables, requires less on-line computer time for measurement, or estimation, of the required process variables. They also include the proportional feedback, integral feedback, feedforward, and/or setpoint control modes which were present in the high order control law.

The first program uses a modal analysis of the high order system to approximate the states which are not wanted in the reduced order control law, as a function of the process inputs and the states to be retained in the reduced order control law. This approach is developed in detail in Chapter Five. In the second program, a least squares analysis is used to fit a reduced order control law to the high order control law using data which are calculated from the high order control law and a sequence of uniformly distributed random numbers. This approach is discussed in Chapter Six.

A third approach for calculating a control law of reduced order makes use of two of the GEMSCOPE programs already described. A reduced order model can be obtained using the programs outlined in Section 2.2.6 and the optimal control law for this model can be calculated, as discussed in Section 2.2.8.

Each of these three approaches to reduced order control law design are discussed in Chapter Six, where it is shown that the approach using the modal analysis to reduce the high order control law gives the best results when applied to the control of the pilot plant evaporator outlined in Appendix A.

2.2.10 Utilization of a Discrete Control Vector

Provision has been made within GEMSCOPE to carry out time domain simulation using control variables, $\underline{u}$, specified by a discrete time series stored in a computer file. The values for $\underline{u}$ can be supplied by the user or calculated by another program. Nieman [24] developed programs which use quasilinearization and/or linear programming techniques to generate the optimal control policy to drive a process from one state to another in such a manner as to minimize a criterion such as minimum time, or minimum sum of the absolute errors. These programs can be used to generate a discrete series of control values, which can be read into GEMSCOPE and used to generate a time domain response.

2.3 DISCUSSION

The programs discussed above form a useful package of simulation and design programs for linear systems. However, there is

still a need for further development of control and numerical techniques that will make the package more flexible and will extend the applications to larger systems, non-linear systems, stochastic systems, etc. Some possible future additions to GEMSCOPE include:

a) Programs to design state vector observers and filters. This was studied by Hamilton [13] and could include the Luenberger observer [16] and the Kalman Filter [14].

b) Altering the time domain response program in conjunction with the design of state estimators, to include the provisions for filtering measurements and for adding random noise to inputs and/or states so noisy processes could be simulated.

c) Programs to include time delays in the multivariable feedback control law, as was studied by Alevisakis [4].

d) Programs to calculate an adaptive control algorithm along with the calculation of the time domain response of the system. One such algorithm was considered by Oliver [26].

e) Programs to design multivariable control laws by techniques other than the dynamic programming approach described above. Some possible schemes include closed-loop pole placement and the newer frequency domain techniques described by MacFarlane [17].

f) A general numerical linearization program which could calculate the linear state space model in the form of Equation (2.1) from a non-linear model provided by the user in the form of a subroutine.

2.4 CONCLUSIONS

GEMSCOPE forms the base of an easily expandable package of simulation and design programs for linear systems. It is designed to facilitate the addition, modification or deletion of individual programs and to make full use of the support programs supplied as part of the computer operating system. Several possible additions to GEMSCOPE have been discussed.

CHAPTER THREE

MODEL REDUCTION FOR DISCRETE-TIME DYNAMIC SYSTEMS

ABSTRACT

The model reduction techniques of Marshall, Davison and Fossard for linear continuous-time models are extended for use in reducing the order of linear discrete-time models. Two approaches for the reduction of high order continuous-time models to low order discrete-time models are presented and evaluated. The resulting reduced order, discrete-time models obtained by the two approaches are shown to be equivalent. A simple numerical example is solved to demonstrate the techniques presented.

3.1 INTRODUCTION

Linear state space models have been extensively used to mathematically describe a wide variety of physical systems, such as aircraft, chemical plants, and refineries. In many situations, dynamic models consisting of a large number of differential or difference equations can be derived from theoretical considerations. Often, such models are so large as to be inconvenient or impractical for many purposes, including simulation and control system design and implementation. Consequently, a need exists for systematic procedures for deriving reduced order dynamic models from high order models.

Model reduction techniques for continuous-time, state space models have been the subject of many recent investigations with modal approaches by Marshall [14], Davison [9], Ghidambara and Davison [5,, 6,7], Graham [12] and Fossard [10] receiving the most attention. By contrast, only a few investigations have considered the model reduction of discrete-time systems. Anderson [4] presented a method for deriving a discrete-time reduced order system based on a least squares fit of the response of the reduced order model to that of the original high order model. He has also presented revisions to his method to reduce computational difficulties (Anderson [3]), to account for pre-specified elements of the reduced order system (Anderson [2]) and to produce better agreement at the final steady state (Anderson [1]). In this investigation the modal methods of Marshall [14], Davison [9] and Fossard [10] are extended to discrete-time models.

The discrete-time model reduction techniques will be derived in Section 3.2. In Section 3.3, the problem of reducing a high order

continuous-time model to a low order discrete-time model is considered. Finally, a numerical example illustrating the model reduction techniques is presented in Section 3.4.

3.2 MODEL REDUCTION FOR DISCRETE-TIME SYSTEMS

3.2.1 Problem Formulation

Consider the stable, time-invariant, discrete-time model;

$$\underline{x}(j+1) = \underline{\underline{\Phi}}\, \underline{x}(j) + \hat{\underline{\underline{\Delta}}}\, \hat{\underline{u}}(j) \tag{3.1}$$

which can be partitioned as:

$$\begin{bmatrix} \underline{x}_1(j+1) \\ \underline{x}_2(j+1) \end{bmatrix} = \begin{bmatrix} \underline{\underline{\Phi}}_1 & \underline{\underline{\Phi}}_2 \\ \underline{\underline{\Phi}}_3 & \underline{\underline{\Phi}}_4 \end{bmatrix} \begin{bmatrix} \underline{x}_1(j) \\ \underline{x}_2(j) \end{bmatrix} + \begin{bmatrix} \hat{\underline{\underline{\Delta}}}_1 \\ \hat{\underline{\underline{\Delta}}}_2 \end{bmatrix} \hat{\underline{u}}(j) \tag{3.2}$$

where $\underline{x}$ = n-dimensional state vector.

$\underline{x}_1$ = ℓ-dimensional vector of states to be retained in the reduced order model.

$\underline{x}_2$ = (n-ℓ)-dimensional vector of states to be eliminated.

$\hat{\underline{u}}$ = (m+q)-dimensional vector of inputs which will include the m-dimensional control vector and the q-dimensional disturbance vector.

$\underline{\underline{\Phi}}$ and $\hat{\underline{\underline{\Delta}}}$ = constant coefficient matrices of appropriate dimensions.

j = a counter for time intervals such that $\underline{x}(j)$ denotes $\underline{x}(t)$ at $t = jT$, where T is the discrete time interval.

The objective is to determine a reduced order model of specified order, ℓ, which is of the form

$$\underline{x}_1(j+1) = \underline{\underline{\Phi}}_R\, \underline{x}_1(j) + \hat{\underline{\underline{\Delta}}}_R\, \hat{\underline{u}}(j) \tag{3.3}$$

where $\underline{\underline{\Phi}}_R$ and $\hat{\underline{\underline{\Delta}}}_R$ are constant matrices of appropriate dimensions and must be determined.

Since the model reduction techniques to be considered are based on a modal analysis, the required preliminary material will be presented.

As n x n matrix, $\underline{\underline{M}}$, exists which transforms Equation (3.1) into the Jordan canonical form (Gantmacher [11]). Define,

$$\underline{x}(j) = \underline{\underline{M}}\, \underline{z}(j) \tag{3.4}$$

or in partitioned form,

$$\begin{bmatrix} \underline{x}_1(j) \\ \underline{x}_2(j) \end{bmatrix} = \begin{bmatrix} \underline{\underline{M}}_1 & \underline{\underline{M}}_2 \\ \underline{\underline{M}}_3 & \underline{\underline{M}}_4 \end{bmatrix} \begin{bmatrix} \underline{z}_1(j) \\ \underline{z}_2(j) \end{bmatrix} \tag{3.5}$$

where $\underline{z}$ is the canonical state vector, $\underline{z}_1$ is an ℓ-dimensional vector, and $\underline{z}_2$ is an $(n-\ell)$-dimensional vector. It is assumed that matrix $\underline{\underline{M}}$ is arranged so that its columns are ordered, from left to right, in order of decreasing significance of the corresponding eigenvalues of $\underline{\underline{\Phi}}$. (This can always be achieved by appropriate column interchange.) Since the system in Equation (3.1) is assumed to be stable, all the eigenvalues of $\underline{\underline{\Phi}}$ lie in the unit circle (Kuo, [13]). Thus, the columns of $\underline{\underline{M}}$ on the left correspond to the eigenvalues of $\underline{\underline{\Phi}}$ which affect $\underline{x}_1$ the most, and those on the right correspond to eigenvalues of $\underline{\underline{\Phi}}$ which affect $\underline{x}_1$ the least. If matrix $\underline{\underline{\Phi}}$ has n distinct eigenvalues, the columns of $\underline{\underline{M}}$ are then the eigenvectors of $\underline{\underline{\Phi}}$ and $\underline{\underline{M}}$ is referred to as the Modal Matrix. When Equation (3.5) is substituted into Equation (3.2), the Jordan canonical form results:

$$\begin{bmatrix} \underline{z}_1(j+1) \\ \underline{z}_2(j+1) \end{bmatrix} = \begin{bmatrix} \underline{\alpha}_1 & \underline{0} \\ \underline{0} & \underline{\alpha}_2 \end{bmatrix} \begin{bmatrix} \underline{z}_1(j) \\ \underline{z}_2(j) \end{bmatrix} + \begin{bmatrix} \hat{\underline{\delta}}_1 \\ \hat{\underline{\delta}}_2 \end{bmatrix} \underline{u}(j) \tag{3.6}$$

where $\underline{\alpha}$, a block diagonal matrix, and $\hat{\underline{\delta}}$, are defined as

$$\underline{\alpha} = \begin{bmatrix} \underline{\alpha}_1 & \underline{0} \\ \underline{0} & \underline{\alpha}_2 \end{bmatrix} = \underline{M}^{-1} \underline{\Phi} \underline{M} \tag{3.7}$$

$$\hat{\underline{\delta}} = \begin{bmatrix} \hat{\underline{\delta}}_1 \\ \hat{\underline{\delta}}_2 \end{bmatrix} = \underline{V} \hat{\underline{\Delta}} \tag{3.8}$$

and $\underline{V}$ is defined as,

$$\underline{V} = \begin{bmatrix} \underline{V}_1 & \underline{V}_2 \\ \underline{V}_3 & \underline{V}_4 \end{bmatrix} = \underline{M}^{-1} \tag{3.9}$$

Because of the way the columns of $\underline{M}$ have been arranged, the Jordan canonical matrix, $\underline{\alpha}$, has the eigenvalues of $\underline{\Phi}$ which most affect $\underline{x}_1$ in the $\ell x \ell$ matrix, $\underline{\alpha}_1$, and those which least affect $\underline{x}_1$ in the $n-\ell$ x $n-\ell$ matrix, $\underline{\alpha}_2$. Hence, it follows from Equation (3.6), that $\underline{z}_1$ represents the modes which most affect $\underline{x}_1$, and $\underline{z}_2$ represents the modes which least affect $\underline{x}_1$. It is also apparent from Equation (3.6) that the modes represented by $\underline{z}_1$ and $\underline{z}_2$ are noninteracting. For systems with n linearly independent eigenvectors, $\underline{\alpha}$ is a diagonal matrix and $\underline{z}_1$ and $\underline{z}_2$ will always be noninteracting. However, for systems with less than n linearly independent eigenvectors, the partitioning of the original system in Equation (3.2) must be such that it does not split a Jordan block in $\underline{\alpha}$, so that $\underline{z}_1$ and $\underline{z}_2$ will remain non-interacting.

3.2.2 Marshall's Analysis

In Marshall's analysis for continuous-time systems (Marshall [14]), the reduced model is derived by neglecting the dynamics of the modes of the original high order model represented by $\underline{z}_2$. This approach retains the significant system modes and offers the advantages of simplicity and zero steady state error for constant inputs. Graham [12] has shown that the model reduction method developed independently by Chidambara (Chidambara and Davison [6]) is equivalent to Marshall's method (Marshall [14]). In this section, Marshall's analysis will be extended to the reduction of discrete-time models.

Equation (3.2) implies the relation,

$$\underline{x}_1(j+1) = \underline{\underline{\Phi}}_1 \underline{x}_1(j) + \underline{\underline{\Phi}}_2 \underline{x}_2(j) + \hat{\underline{\underline{\Delta}}}_1 \hat{\underline{u}}(j) \tag{3.10}$$

An expression for $\underline{x}_2(j)$ can be obtained by combining Equations (3.4) and (3.9) to give

$$\underline{x}_2(j) = \underline{\underline{V}}_4^{-1} (\underline{z}_2(j) - \underline{\underline{V}}_3 \underline{x}_1(j)) \tag{3.11}$$

As an approximation, assume that $\underline{z}_2(j)$ immediately attains the new steady state value (i.e., instantaneous response). Since $\underline{z}_2$ does not affect $\underline{x}_1$ significantly, the effect of this approximation on $\underline{x}_1$ is small. Thus, from Equation (3.6),

$$\underline{z}_2(j+1) = \underline{z}_2(j) = (\underline{\underline{I}} - \underline{\underline{\alpha}}_2)^{-1} \hat{\underline{\underline{\delta}}}_2 \hat{\underline{u}}(j) \tag{3.12}$$

Now, combining Equations (3.10), (3.11) and (3.12) gives the reduced order system of Equation (3.3) with,

$$\underline{\Phi}_R = \underline{\Phi}_1 - \underline{\Phi}_2 \underline{V}_4^{-1} \underline{V}_3 \tag{3.13}$$

$$\hat{\underline{\Delta}}_R = \hat{\underline{\Delta}}_1 + \underline{\Phi}_2 \underline{V}_4^{-1} (\underline{I} - \underline{\alpha}_2)^{-1} \hat{\underline{\delta}}_2 \tag{3.14}$$

Thus, Marshall's analysis for continuous-time models can be extended to provide a straightforward method for reducing the order of discrete-time models. For constant inputs, the reduced and original systems reach the same steady state value of $\underline{x}_1$. For other types of sustained inputs (e.g., a ramp or a sinusoid) there is a continuing error between the responses of the reduced and original systems (Graham [12]). However, this error is often small.

3.2.3 Davison's Analysis

Davison's analysis for continuous-time systems (Davison [9]) is based on the assumption that the contribution of the insignificant eigenvalues of the original system to the system response is small and may be neglected. In Davison's analysis, a reduced system results in which the dominant modes are excited in the same proportion as in the original system.

The solution to Equation (3.1) can be written as,

$$\underline{x}(j+1) = \underline{\Phi}^{j+1} \underline{x}(0) + \left\{ \sum_{k=0}^{j} \underline{\Phi}^{j-k} \hat{\underline{\Delta}} \hat{\underline{u}}(k) \right\} \tag{3.15}$$

where $\underline{x}(0)$ is the initial condition for vector $\underline{x}$. Following Davison [9], consider the case with $\underline{x}(0) = \underline{0}$. Then,

$$\underline{x}(j+1) = \left\{ \sum_{k=0}^{j} \underline{\Phi}^{j-k} \hat{\underline{\Delta}} \hat{\underline{u}}(k) \right\} \tag{3.16}$$

Using the theorem in Appendix D.1 and Equation (3.7), Equation (3.16) can be written as,

$$\underline{x}(j+1) = \underline{M} \sum_{k=0}^{j} \underline{\alpha}^{j-k} \underline{M}^{-1} \hat{\underline{\Delta}} \hat{\underline{u}}(k) \tag{3.17}$$

Equation (3.17) becomes, after partitioning,

$$\begin{bmatrix} \underline{x}_1(j+1) \\ \underline{x}_2(j+1) \end{bmatrix} = \begin{bmatrix} \underline{M}_1 & \underline{M}_2 \\ \underline{M}_3 & \underline{M}_4 \end{bmatrix} \left\{ \sum_{k=0}^{j} \left\{ \begin{bmatrix} \underline{\alpha}_1^{j-k} & \underline{0} \\ \underline{0} & \underline{\alpha}_2^{j-k} \end{bmatrix} \begin{bmatrix} \underline{V}_1 & \underline{V}_2 \\ \underline{V}_3 & \underline{V}_4 \end{bmatrix} \hat{\underline{\Delta}} \hat{\underline{u}}(k) \right\} \right\} \tag{3.18}$$

As an approximation, assume that the eigenvalues in $\underline{\alpha}_2$ have no effect on the system response, and therefore can be neglected (i.e., $\underline{\alpha}_2 = \underline{0}$). Equation (3.18) becomes,

$$\begin{bmatrix} \underline{x}_1(j+1) \\ \underline{x}_2(j+1) \end{bmatrix} = \begin{bmatrix} \underline{M}_1 \\ \underline{M}_3 \end{bmatrix} \left\{ \sum_{k=0}^{j} \left\{ \underline{\alpha}_1^{j-k} (\underline{V}_1, \underline{V}_2) \hat{\underline{\Delta}} \hat{\underline{u}}(k) \right\} \right\} \tag{3.19}$$

Define an ℓ x 1 vector $\underline{\xi}$ as

$$\underline{\xi} = \sum_{k=0}^{j} \left\{ \underline{\alpha}_1^{j-k} (\underline{V}_1, \underline{V}_2) \hat{\underline{\Delta}} \hat{\underline{u}}(k) \right\}$$

and Equation (3.19) becomes,

$$\begin{bmatrix} \underline{x}_1(j+1) \\ \underline{x}_2(j+1) \end{bmatrix} = \begin{bmatrix} \underline{M}_1 \\ \underline{M}_3 \end{bmatrix} \underline{\xi} \tag{3.20}$$

Rearranging Equation (3.20) gives

$$\underline{\xi} = \underline{M}_1^{-1} \underline{x}_1(j+1)$$

and

$$\underline{x}_2(j+1) = \underline{M}_3 \underline{M}_1^{-1} \underline{x}_1(j+1) \tag{3.21}$$

As in Davison [9], $\underline{\underline{\Phi}}_R$ and $\hat{\underline{\underline{\Delta}}}_R$ are calculated by the following analysis. Equation (3.21) is substituted into the unforced form of Equation (3.10) (i.e., $\hat{\underline{u}}(j) = \underline{0}$) to give,

$$\underline{x}_1(j+1) = (\underline{\underline{\Phi}}_1 + \underline{\underline{\Phi}}_2 \underline{\underline{M}}_3 \underline{\underline{M}}_1^{-1}) \underline{x}_1(j)$$

so that,

$$\underline{\underline{\Phi}}_R = \underline{\underline{\Phi}}_1 + \underline{\underline{\Phi}}_2 \underline{\underline{M}}_3 \underline{\underline{M}}_1^{-1} \tag{3.22}$$

To calculate $\hat{\underline{\underline{\Delta}}}_R$, the solutions of the original model and the reduced order model are compared with $\underline{x}(0) = \underline{0}$.

The solution of the original model in Equation (3.1) for $\underline{x}_1$, ignoring $\underline{\alpha}_2$, is,

$$\underline{x}_1(j+1) = \underline{\underline{M}}_1 \left\{ \sum_{k=0}^{j} \left\{ \underline{\underline{\alpha}}_1^{j-k} (\underline{\underline{V}}_1 \hat{\underline{\underline{\Delta}}}_1 + \underline{\underline{V}}_2 \hat{\underline{\underline{\Delta}}}_2) \hat{\underline{u}}(k) \right\} \right\} \tag{3.23}$$

The solution of the reduced order model is,

$$\underline{x}_1(j+1) = \sum_{k=0}^{j} \left\{ \underline{\underline{\Phi}}_R^{j-k} \hat{\underline{\underline{\Delta}}}_R \hat{\underline{u}}(k) \right\}$$

However, using the equations, $\underline{\underline{\Phi}} = \underline{\underline{M}} \underline{\underline{\alpha}} \underline{\underline{V}}$ and $\underline{\underline{V}} \underline{\underline{M}} = \underline{\underline{I}}$, it can be shown that

$$\underline{\underline{\Phi}}_R = \underline{\underline{M}}_1 \underline{\underline{\alpha}}_1 \underline{\underline{M}}_1^{-1}$$

so this reduced order solution becomes,

$$\underline{x}_1(j+1) = \underline{\underline{M}}_1 \left\{ \sum_{k=0}^{j} \left\{ \underline{\underline{\alpha}}_1^{j-k} \underline{\underline{M}}_1^{-1} \hat{\underline{\underline{\Delta}}}_R \hat{\underline{u}}(k) \right\} \right\} \tag{3.24}$$

Equating Equations (3.23) and (3.24) gives the following expression for $\hat{\underline{\underline{\Delta}}}_R$.

$$\hat{\underline{\Delta}}_R = \underline{M}_1 \left(\underline{V}_1 \hat{\underline{\Delta}}_1 + \underline{V}_2 \hat{\underline{\Delta}}_2\right) = \underline{M}_1 \hat{\underline{\delta}}_1 \tag{3.25}$$

Thus, a simple, reduced order, discrete-time system has been derived using Davison's modal approach. This analysis could also be extended to include the various modifications of Davison's Method (Davison [8], Chidambara and Davison [6]).

It can be shown, using the partitioned form of the expression $\underline{VM} = \underline{I}$, that the transition matrix in Equation (3.22) is equivalent to the transition matrix in Equation (3.13), obtained by Marshall's analysis.

3.2.4 Fossard's Analysis

Fossard [10] presented a revision to Davison's continuous-time result to correct the steady state error which results between the reduced order model and the original high order model after a step change in the input. This correction is developed here for the discrete-time form of the model.

Consider the value of $\underline{x}_1(j)$ from Equation (3.5) as

$$\underline{x}_1(j) = \underline{M}_1 \underline{z}_1(j) + \underline{M}_2 \underline{z}_2(j) \tag{3.26}$$

The steady state values of $\underline{z}_1$ and $\underline{z}_2$ are calculated from Equation (3.6) as

$$\underline{z}_{1ss} = (\underline{I}-\underline{\alpha}_1)^{-1} \hat{\underline{\delta}}_1 \hat{\underline{u}}_{ss} \tag{3.27}$$

$$\underline{z}_{2ss} = (\underline{I}-\underline{\alpha}_2)^{-1} \hat{\underline{\delta}}_2 \hat{\underline{u}}_{ss} \tag{3.28}$$

Thus, the steady state $\underline{x}_1$ vector can be expressed as

$$\underline{x}_{1ss} = \underline{M}_1(\underline{I}-\underline{\alpha}_1)^{-1} \hat{\underline{\delta}}_1 \hat{\underline{u}}_{ss} + \underline{M}_2(\underline{I}-\underline{\alpha}_2)^{-1} \hat{\underline{\delta}}_2 \hat{\underline{u}}_{ss} \tag{3.29}$$

The steady state vector, $\underline{x}_{1ss}$, evaluated using Davison's resulting reduced order model can be obtained in a similar manner

$$\underline{x}_1(j) = \underline{M}_1 \underline{z}_1(j) \tag{3.30}$$

$$\underline{z}_{1ss} = (\underline{I}-\underline{\alpha}_1)^{-1} \hat{\underline{\delta}}_1 \hat{\underline{u}}_{ss} \tag{3.31}$$

Thus,

$$\underline{x}_{1ss} = \underline{M}_1 (\underline{I}-\underline{\alpha}_1)^{-1} \hat{\underline{\delta}}_1 \hat{\underline{u}}_{ss} \tag{3.32}$$

To have the steady state in Equation (3.32) agree with that in Equation (3.29) the term

$$\underline{M}_2 (\underline{I}-\underline{\alpha}_2)^{-1} \hat{\underline{\delta}}_2 \hat{\underline{u}}(j)$$

must be added to Equation (3.32). This is accomplished in the following reduced order discrete-time model which corresponds to the continuous-time result presented by Fossard.

$$\underline{x}_\ell(j+1) = \underline{\Phi}_R \underline{x}_\ell(j) + \hat{\underline{\Delta}}_R \hat{\underline{u}}(j) \tag{3.33}$$

$$\underline{x}_1(j) = \underline{x}_\ell(j) + \underline{E}_R \hat{\underline{u}}(j) \tag{3.34}$$

where

$$\underline{E}_R = \underline{M}_2 (\underline{I}-\underline{\alpha}_2)^{-1} \hat{\underline{\delta}}_2 \tag{3.35}$$

$\underline{\Phi}_R$ and $\hat{\underline{\Delta}}_R$ = the results of Davison's analysis and are given in Equations (3.22) and (3.25) respectively.

3.3 REDUCTION OF HIGH ORDER CONTINUOUS MODELS TO LOW ORDER DISCRETE MODELS

Models of physical systems are often represented as large sets of linear differential equations, in the general form,

$$\dot{\underline{x}} = \underline{\underline{A}}\underline{x} + \hat{\underline{\underline{B}}}\hat{\underline{u}} \tag{3.36}$$

where $\underline{x}$ and $\hat{\underline{u}}$ are defined in Section 3.2.1, and $\underline{\underline{A}}$ and $\hat{\underline{\underline{B}}}$ are matrices of appropriate dimensions. For both simulation studies and the design and implementation of computer control systems, it may be desirable to have available a low order discrete-time model.

Two general approaches for deriving a low order discrete-time model from a high order continuous-time model are shown in Figure 3.1. The high order model can first be reduced and then the resulting low order model discretized (Approach I), or alternatively, the high order model can be discretized and this discrete-time model can then be reduced (Approach II). The continuous-time, reduced order methods of Marshall [14], Davison [9] and Fossard [10] will be needed for Approach I and will now be summarized.

The transformation matrix $\underline{\underline{M}}$ exists (see the theorem in Appendix D.1) which transforms Equation (3.36) into the Jordan canonical form as,

$$\underline{x} = \underline{\underline{M}}\underline{z} \tag{3.37}$$

$$\dot{\underline{z}} = \underline{\underline{J}}\underline{z} + \hat{\underline{\underline{G}}}\hat{\underline{u}} = \begin{bmatrix} \underline{\underline{J}}_1 & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{J}}_2 \end{bmatrix} \begin{bmatrix} \underline{z}_1 \\ \underline{z}_2 \end{bmatrix} + \begin{bmatrix} \hat{\underline{\underline{G}}}_1 \\ \hat{\underline{\underline{G}}}_2 \end{bmatrix} \hat{\underline{u}} \tag{3.38}$$

where

$$\underline{\underline{J}} = \underline{\underline{M}}^{-1}\underline{\underline{A}}\underline{\underline{M}} \tag{3.39}$$

$$\hat{\underline{\underline{G}}} = \underline{\underline{M}}^{-1}\hat{\underline{\underline{B}}} = \underline{\underline{V}}\hat{\underline{\underline{B}}} \tag{3.40}$$

To calculate the continuous-time reduced order model, the high order model in Equation (3.26) and the transformation matrix $\underline{\underline{M}}$ must be arranged and partitioned as was discussed in Section 3.2.1.

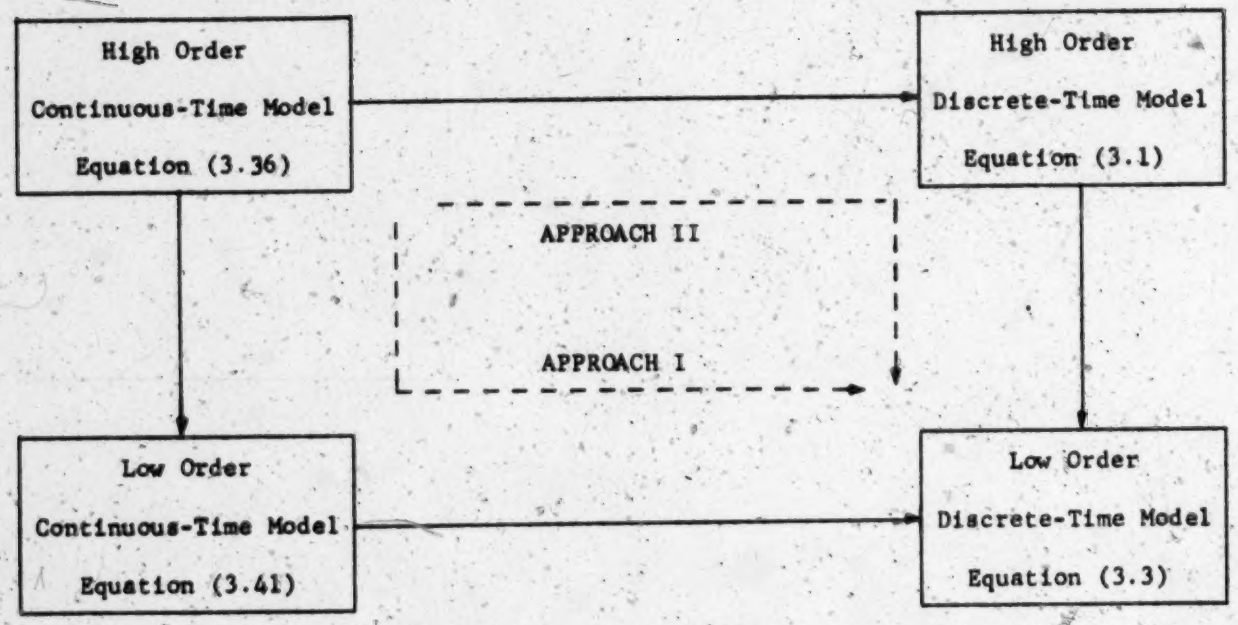


FIGURE 3.1 TWO APPROACHES FOR REDUCING A HIGH ORDER CONTINUOUS-TIME MODEL TO A LOW ORDER DISCRETE-TIME MODEL

The reduced-order model is of the form,

$$\dot{\underline{x}}_1 = \underline{\underline{A}}_R \underline{x}_1 + \hat{\underline{\underline{B}}}_R \hat{\underline{u}} \tag{3.41}$$

where Marshall's method [14] gives

$$\underline{\underline{A}}_R = \underline{\underline{M}}_1 \underline{\underline{J}}_1 \underline{\underline{M}}_1^{-1} \tag{3.42}$$

$$\hat{\underline{\underline{B}}}_R = \hat{\underline{\underline{B}}}_1 - \underline{\underline{A}}_2 \underline{\underline{V}}_4 \underline{\underline{J}}_2^{-1} \hat{\underline{\underline{G}}}_2 \tag{3.43}$$

and Davison's method [9] yields,

$$\underline{\underline{A}}_R = \underline{\underline{A}}_1 + \underline{\underline{A}}_2 \underline{\underline{M}}_3 \underline{\underline{M}}_1^{-1} \tag{3.44}$$

$$\hat{\underline{\underline{B}}}_R = \underline{\underline{M}}_1 \hat{\underline{\underline{G}}}_1 \tag{3.45}$$

Fossard's method produces a reduced order model in the form

$$\dot{\underline{x}}_\ell = \underline{\underline{A}}_R \underline{x}_\ell + \hat{\underline{\underline{B}}}_R \hat{\underline{u}} \tag{3.46}$$

$$\underline{x}_1 = \underline{x}_\ell + \underline{\underline{E}}_R \hat{\underline{u}} \tag{3.47}$$

where $\underline{\underline{A}}_R$ and $\hat{\underline{\underline{B}}}_R$ are given in Equations (3.44) and (3.45) and

$$\underline{\underline{E}}_R = -\underline{\underline{M}}_2 \underline{\underline{J}}_2^{-1} \hat{\underline{\underline{G}}}_2 \tag{3.48}$$

It is shown in Appendix B.3 that the discrete-time form of $\underline{\underline{E}}_R$ presented in Equation (3.35) is identical to the continuous-time form presented in Equation (3.48). This is expected since Equations (3.34) and (3.47) are identical in form and must hold for all time.

Approaches I and II differ in the order in which the individual steps of model reduction and discretization are carried out. From the nature of these operations, intuitively one expects that both paths will result in an identical reduced order discrete-time model.

This equivalence is shown analytically in Appendices B.1, B.2 and B.3 by making use of the definitions of $\underline{\Phi}$ and $\hat{\underline{\Delta}}$ in terms of $\underline{A}$ and $\hat{\underline{B}}$ as,

$$\underline{\Phi} = \exp(\underline{A}T)$$

$$\hat{\underline{\Delta}} = \int_0^T \exp(\underline{A}(T-\tau))\hat{\underline{B}}d\tau \tag{3.49}$$

and $\underline{\Phi}_R$ and $\hat{\underline{\Delta}}_R$ in terms of $\underline{A}_R$ and $\hat{\underline{B}}_R$ as,

$$\underline{\Phi}_R = \exp(\underline{A}_R T)$$

$$\hat{\underline{\Delta}}_R = \int_0^T \exp(\underline{A}_R(T-\tau))\hat{\underline{B}}_R d\tau \tag{3.50}$$

The equivalence of Approaches I and II will also be demonstrated by the numerical example in Section 3.4.

Since the order of model reduction and discretization does not affect the resulting reduced order discrete-time model, one should consider which approach is the most convenient for a particular application. If the only requirement is the derivation of a low order discrete-time model from the original high order continuous-time model, then Approach I should be followed, since its computational requirements are less. The continuous-time reduction and the discrete-time reduction involve the same number of matrix operations. However, the derivation of the high order discrete-time system using Equation (3.49) requires many more calculations than the derivation of the low order discrete-time system in Equation (3.50). Thus, Approach I is computationally more efficient than Approach II. However, if the availability of a high order, discrete-time model is desirable, perhaps for the evalua-

tion of proposed control systems, Approach II should be used since the high order, discrete-time model must be derived in any event.

It is interesting to compare the results of the discrete-time model reduction with the results of the continuous-time reduction. It can be shown by elementary matrix algebra, using the partitioned forms of the expressions $\underline{\Phi} = \underline{M}\ \underline{\alpha}\ \underline{V}$ and $\underline{MV} = \underline{I}$, that the discrete transition matrix in Equation (3.13) resulting from Marshall's analysis is equivalent to the following,

$$\underline{\Phi}_R = \underline{\Phi}_1 - \underline{\Phi}_2 \underline{V}_4^{-1} \underline{V}_3 = \underline{M}_1 \underline{\alpha}_1 \underline{M}_1^{-1} \tag{3.51}$$

Thus, by comparing Equation (3.51) with (3.42), Equation (3.14) with (3.43), Equation (3.22) with (3.44) and Equation (3.25) with (3.45), it can be seen that there is a direct, term-by-term correspondence between the discrete-time and the continuous-time results.

3.4 EXAMPLES

3.4.1 Second Order Example

A simple problem considered by Chidambara and Davison [6] and by Graham [12], will be used to demonstrate both the discrete-time model reduction techniques and the equivalence of the two approaches shown in Figure 3.1. Consider the second order continuous-time system in Equation (3.52), which is to be reduced to a first order discrete-time system in $\underline{x}_1$.

$$\dot{\underline{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{3.52}$$

The eigenvalues of Equation (3.52) are -1 and -4 so that,

$$\underline{J} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$$

and

$$\underline{M} = \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 4/3 \end{bmatrix}, \quad \underline{V} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$$

The equivalent discrete-time system, with sampling period T is

$$\underline{x}(j+1) = 1/3 \begin{bmatrix} 4e^{-T} - e^{-4T}, & e^{-T} - e^{-4T} \\ -4e^{-T} + 4e^{-4T}, & -e^{-T} + 4e^{-4T} \end{bmatrix} \underline{x}(j) + 1/3 \begin{bmatrix} 3/4 - e^{-T} + 1/4e^{-4T} \\ e^{-T} - e^{-4T} \end{bmatrix} u(j) \tag{3.53}$$

Equation (3.53) has eigenvalues of e^{-T} and e^{-4T}.

Marshall's Analysis

Marshall's analysis for Approach I, using Equations (3.42) and (3.43) gives,

$$\underline{A}_R = \underline{M}_1 \underline{J}_1 \underline{M}_1^{-1} = 1$$

$$\hat{\underline{B}}_R = \hat{\underline{B}}_1 - \underline{A}_2 \underline{V}_4^{-1} \underline{J}_2^{-1} \hat{\underline{G}}_2 = 1/4$$

Thus, the low order continuous-time model is

$$\dot{x}_1 = -x_1 + u/4 \tag{3.54}$$

with the corresponding discrete-time model as

$$x_1(j+1) = e^{-T} x_1(j) + (1-e^{-T})\, u(j)/4 \tag{3.55}$$

For Approach II, using Equations (3.13), (3.14), and (3.53)

$$\underline{\Phi}_R = \underline{\Phi}_1 - \underline{\Phi}_2 \underline{V}_4^{-1} \underline{V}_3 = e^{-T}$$

$$\hat{\underline{\Delta}}_R = \hat{\underline{\Delta}}_1 + \underline{\Phi}_2 \underline{V}_4^{-1} (\underline{I} - \underline{\alpha}_2)^{-1} \hat{\underline{\delta}}_2 = (1-e^{-T})/4$$

and the reduced order, discrete-time model from Approach II is,

$$x_1(j+1) = e^{-T} x_1(j) + (1-e^{-T})u(j)/4 \qquad (3.56)$$

which is identical to Equation (3.55).

Davison's Analysis

Davison's analysis, for Approach I, using Equations (3.44) and (3.45) gives,

$$\underline{A}_R = \underline{A}_1 + \underline{A}_2 \underline{M}_3 \underline{M}_1^{-1} = -1$$

$$\hat{\underline{B}}_R = \underline{M}_1 \hat{\underline{G}}_1 = 1/3$$

The low order, continuous-time model is,

$$\dot{x}_1 = -x_1 + u/3 \qquad (3.57)$$

with the corresponding discrete-time model as

$$x_1(j+1) = e^{-T} x_1(j) + (1-e^{-T})\, u(j)/3 \qquad (3.58)$$

For Approach II, using Equations (3.22), (3.25), and (3.53)

$$\underline{\Phi}_R = \underline{\Phi}_1 + \underline{\Phi}_2 \underline{M}_3 \underline{M}_1^{-1} = e^{-T}$$

$$\hat{\underline{\Delta}}_R = \underline{M}_1 (\underline{V}_1 \hat{\underline{\Delta}}_1 + \underline{V}_2 \hat{\underline{\Delta}}_2) = (1-e^{-T})/3$$

and the reduced order, discrete-time model from Approach II is,

$$x_1(j+1) = e^{-T} x_1(j) + (1-e^{-T})\, u(j)/3 \qquad (3.59)$$

which is identical to Equation (3.58).

Fossard's Analysis

Fossard's analysis produces the same $\underline{\underline{A}}_R$, $\hat{\underline{\underline{B}}}_R$, $\underline{\underline{\Phi}}_R$ and $\hat{\underline{\underline{\Delta}}}_R$ as does Davison's analysis above. $\underline{\underline{E}}_R$ can be calculated using two results. The discrete-time result can be obtained using Equation (3.35) as

$$\underline{\underline{E}}_R = \underline{\underline{M}}_2 (\underline{\underline{I}}-\underline{\underline{\alpha}}_2)^{-1} \hat{\underline{\underline{\delta}}}_2 = -1/12$$

and the continuous-time result, Equation (3.48), produces

$$\underline{\underline{E}}_R = -\underline{\underline{M}}_2 \underline{\underline{J}}_2^{-1} \hat{\underline{\underline{G}}}_2 = -1/12$$

These are identical results as is expected.

3.4.2 Evaporator Model

As a check of the computer program, the tenth order continuous-time evaporator model presented in Appendix A was reduced to a third order discrete-time model by the two approaches of Figure 3.1 using Marshall's analysis. In both cases W1, W2 and C2 were the states retained in the reduced order model and $\underline{x}_1 = [W1, W2, C2]^T$. The eigenvalues retained in the reduced order model were the three closest to zero in the continuous-time case and the three closest to one in the discrete-time case. Figure 3.2 shows that the response of the two resulting third order models are identical when forced by a +20% change in feed flow rate. The third order evaporator models calculated by Marshall's analysis are shown in Table 3.1, where the similarity of the two discrete-time results can be observed.

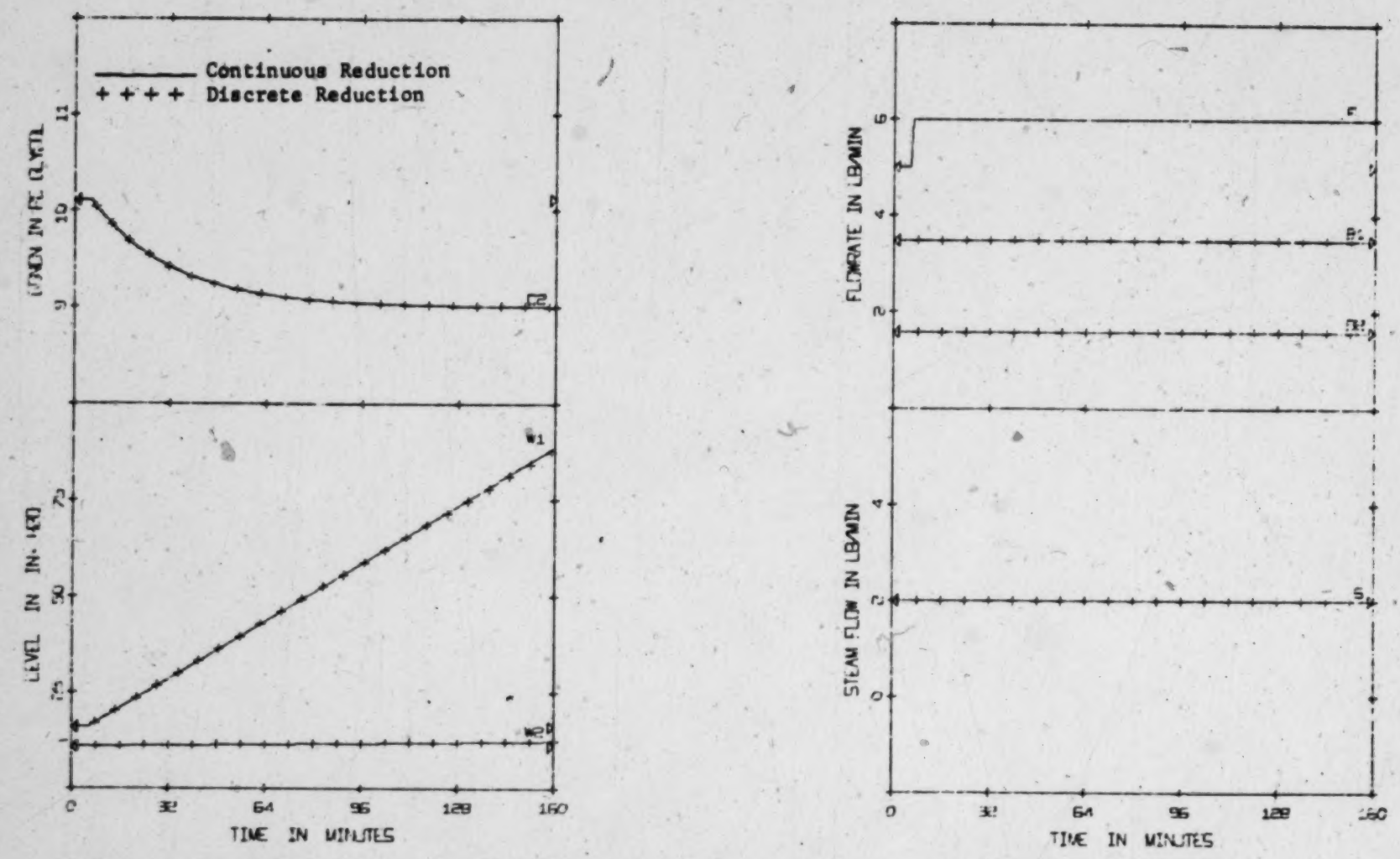


FIGURE 3.2 RESPONSE OF THIRD ORDER MODELS CALCULATED BY CONTINUOUS AND DISCRETE MODAL REDUCTION (3L(2021), 3L(2050)/D, 20%F/OL/2021, 2050)

TABLE 3.1

THIRD ORDER EVAPORATOR MODELS

Continuous-Time Model (Approach I)

$$\underline{A}_R = \begin{bmatrix} -0.4534E-06 & -0.3554E-07 & -0.1532E-06 \\ 0.8203E-07 & 0.2477E-06 & 0.1369E-05 \\ -0.9030E-05 & -0.2938E-04 & -0.3808E-01 \end{bmatrix}$$

$$\underline{B}_R = \begin{bmatrix} 0.1124E\ 00 & -0.9173E-04 & -0.1263E-01 & -0.3050E-01 & -0.7603E-01 & 0.7019E-08 \\ 0.3052E-02 & -0.7374E-04 & -0.1466E-01 & -0.3541E-01 & 0.8007E-01 & -0.3808E-01 \\ -0.2090E-01 & 0.3823E-01 & 0.2095E-01 & 0.5060E-01 & -0.4222E-01 & -0.2524E-07 \end{bmatrix}$$

Discrete-Time Model (Approach I)

$$\underline{\Phi}_R = \begin{bmatrix} 0.9999E\ 00 & -0.3791E-07 & -0.1602E-06 \\ 0.8754E-07 & 0.9999E\ 00 & 0.1431E-05 \\ -0.9439E-05 & -0.3072E-04 & 0.9601E\ 00 \end{bmatrix}$$

$$\underline{\Delta}_R = \begin{bmatrix} 0.1200E\ 00 & -0.9785E-04 & -0.1348E-01 & -0.3254E-01 & -0.8110E-01 & 0.8257E-08 \\ 0.3256E-02 & -0.7864E-04 & -0.1565E-01 & -0.3778E-01 & 0.8540E-01 & -0.4062E-01 \\ -0.2186E-01 & 0.3997E-01 & 0.2191E-01 & 0.5290E-01 & -0.4414E-01 & 0.6018E-06 \end{bmatrix}$$

Discrete-Time Model (Approach II)

$$\underline{\Phi}_R = \begin{bmatrix} 0.9998E\ 00 & -0.3795E-07 & -0.2348E-06 \\ 0.8633E-07 & 0.9998E\ 00 & 0.1662E-05 \\ -0.9408E-05 & -0.3070E-04 & 0.9600E\ 00 \end{bmatrix}$$

$$\underline{\Delta}_R = \begin{bmatrix} 0.1200E\ 00 & -0.9780E-04 & -0.1345E-01 & -0.3249E-01 & -0.8108E-01 & 0.8226E-08 \\ 0.3249E-02 & -0.7866E-04 & -0.1561E-01 & -0.3770E-01 & 0.8539E-01 & -0.4062E-01 \\ -0.2177E-01 & 0.3979E-01 & 0.2183E-01 & 0.5272E-01 & -0.4413E-01 & 0.6023E-06 \end{bmatrix}$$

3.5 CONCLUSIONS

Existing model reduction techniques of Marshall, Davison, and Fossard for continuous-time systems have been extended for use with discrete-time systems. Two approaches for reducing a high order continuous-time model to a low order discrete-time model have been evaluated. The reduction of a continuous-time model followed by discretization is computationally more efficient, but analytically equivalent to, discretizing the high order system and then reducing the high order discrete-time model using the methods outlined in this chapter.

CHAPTER FOUR

MODEL REDUCTION BY LEAST SQUARES USING RANDOM DATA

ABSTRACT

A least squares analysis is applied to model reduction using data which is specified by the high order model and a sequence of uniformly distributed random numbers. This approach produces a reduced order model that is better than the model calculated by the traditional trajectory fitting approach to least squares when applied to the reduction of a tenth order evaporator model. This approach also eliminates some of the difficulties found in the trajectory fitting application of least squares to model reduction.

This least squares approach is applied to the reduction of both the continuous-time and the discrete-time evaporator models.

The problem of calculating a reduced order discrete-time model from a high order continuous-time model is considered. It is shown that, using the same sequence of random numbers, different reduced order discrete-time models will be calculated depending upon whether the model reduction or the model discretization step is calculated first.

4.1 INTRODUCTION

Many physical systems, such as chemical plants, have been mathematically described by linear, state space models, which consist of large numbers of differential or difference equations. These models often contain so many equations that they cannot economically be used for simulation studies or for control system design. Thus, a need exists for procedures to eliminate some of the less important variables from the high order model to obtain a dynamic model of reduced order.

4.1.1 Previous Work

Many different approaches to model reduction have been considered. In the derivation of most complex chemical process models, an intuitive reduction procedure is applied when the modeller assumes that a particular variable responds fast, and so uses a steady state relation. This approach relies upon experience with the particular process being modelled.

Mathematical approaches to model reduction using a modal analysis and a least squares analysis have been considered. Marshall [13], Davison [6], Fossard [7] and others used a modal analysis of the high order continuous-time model in order to calculate reduced order models. These methods were extended for use with discrete-time models in Chapter Three.

As an alternative mathematical approach, Anderson [4] calculated a reduced order, discrete-time model using a least squares analysis, to fit the response of the reduced order model to that of

the original high order model. This approach was extended for the reduction of continuous-time models by Anderson and Kwan [5]. Anderson [3] reformulated his original procedure in order to reduce the computer storage requirements. Anderson [2] also discussed the applications of least squares to model reduction when some of the elements in the reduced order model matrices can be prespecified. In addition, Anderson [1] considers improving the agreement between the high order and reduced order models in specific regions of the response, by adding extra data points in the region of interest. This, however, would be at the expense of poorer agreement in other regions of the response. Graham et al. [8,9] use least squares along with quasi-linearization for the reduction of high order, nonlinear models. They apply a parameter significance test described by Rauch [17] to determine any of the unknown parameters which can be neglected and set equal to zero. Their method is also applicable to linear models.

4.1.2 Difficulties in Previous Least Squares Approach

An important aspect of the use of the least squares approach to model reduction is the type of data which is used. Anderson [4] uses data obtained from a time domain response of the high order model, but he does not indicate for what conditions this data should be generated. As was discussed by Nicholson and Anderson [15], if a time domain response is obtained by input forcing, the inputs must be linearly independent to avoid a linearly dependent set of equations in the least squares solution. Thus, a step in each input cannot be used. Anderson [15] points out that a separate reduced order model

could be calculated for each input, but adds that " ... while such results are not meaningless, they are neither desirable nor useful, ...". McGinnis [14] took this approach in deriving reduced order models of a distillation column. He obtained a different reduced order (2 x 2) transition matrix for each of his five inputs. Then, by trial and error, he was able to find a transition matrix that gave satisfactory results for all his inputs.

If a reduced order model is calculated for good agreement with a particular response caused by one set of input conditions, it would be biased towards that particular set of conditions. The response of the resulting reduced order model would then have poorer agreement with the response of the high order model generated by any other set of inputs.

The following sections of this chapter summarize the least squares approach to model reduction and discuss the use of random numbers as data for the least squares solution, to eliminate the problems discussed in this section.

4.2 LEAST SQUARES WITH RANDOM DATA

The fitting technique used in this work is the classical least squares technique as described in numerical analysis textbooks, for example by Lapidus [12] and as used for model reduction by Anderson [2 - 4]. The only difference between Anderson's work and that presented here, is the method of specifying the data which is used in the least squares fit. The least squares approach will be outlined for the reduction of discrete-time models and will be sum-

marized in Section 4.2.5 for the reduction of continuous-time models.

The linear, time-invariant discrete-time model is in the form

$$\underline{x}(j+1) = \underline{\underline{\Phi}}\,\underline{x}(j) + \hat{\underline{\underline{\Delta}}}\,\hat{\underline{u}}(j) \tag{4.1}$$

where $\underline{x}$ = n-dimensional state vector

$\hat{\underline{u}}$ = (m + q)-dimensional input vector which includes the m-dimensional control vector and the q-dimensional disturbance vector.

$\underline{\underline{\Phi}}$, $\hat{\underline{\underline{\Delta}}}$ = constant coefficient matrices of appropriate dimensions.

The state vector can be partitioned as

$$\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} \tag{4.2}$$

where $\underline{x}_1$ = ℓ-dimensional vector of states to be retained in the reduced order model

$\underline{x}_2$ = $(n-\ell)$-dimensional vector of the remaining states.

The desired reduced order model is in the form of

$$\underline{x}_1(j+1) = \underline{\underline{\Phi}}_R\,\underline{x}_1(j) + \hat{\underline{\underline{\Delta}}}_R\,\hat{\underline{u}}(j) \tag{4.3}$$

where $\underline{\underline{\Phi}}_R$ and $\hat{\underline{\underline{\Delta}}}_R$ are constant coefficient matrices of appropriate dimensions. This section will discuss the evaluation of $\underline{\underline{\Phi}}_R$ and $\hat{\underline{\underline{\Delta}}}_R$ under various conditions. Sections 4.2.1, 4.2.2 and 4.2.3 will deal with the case when the high order system is stable. This is the case

when the eigenvalues of $\underline{\Phi}$ lie between zero and one. Section 4.2.4 will discuss the case where the model contains integrating states corresponding to eigenvalues of one. Section 4.2.5 outlines the results for continuous-time systems while Section 4.2.6 discusses other applications of the theory.

4.2.1 Calculation of $\underline{\Phi}_R$ for a Stable System

To calculate $\underline{\Phi}_R$, consider the following unforced, partitioned forms of Equations (4.1) and (4.3)

$$\begin{bmatrix} \underline{x}_1(j+1) \\ \underline{x}_2(j+1) \end{bmatrix} = \begin{bmatrix} \underline{\Phi}_1 & \underline{\Phi}_2 \\ \underline{\Phi}_3 & \underline{\Phi}_4 \end{bmatrix} \begin{pmatrix} \underline{x}_1(j) \\ \underline{x}_2(j) \end{pmatrix} \tag{4.4}$$

and

$$\underline{x}_1(j+1) = \underline{\Phi}_R \, \underline{x}_1(j) \tag{4.5}$$

An expression for $\underline{\Phi}_R$ can be written as the least squares solution of

$$\underline{W} = \underline{Z} \, \underline{\Phi}_R^T \tag{4.6}$$

That is,

$$\underline{\Phi}_R^T = (\underline{Z}^T \, \underline{Z})^{-1} \, \underline{Z}^T \, \underline{W} \tag{4.7}$$

where

$$\underline{W} = \begin{bmatrix} \underline{w}^T(1) \\ \underline{w}^T(2) \\ \vdots \\ \underline{w}^T(k) \end{bmatrix} \quad \text{and} \quad \underline{Z} = \begin{bmatrix} \underline{x}_1^T(1) \\ \underline{x}_1^T(2) \\ \vdots \\ \underline{x}_1^T(k) \end{bmatrix} \tag{4.8}$$

and where k is the number of data vectors. The j^{th} vector $\underline{w}(j)$ is defined as

$$\underline{w}(j) = \underline{x}_1(j+1) = \underline{\Phi}_1 \underline{x}_1(j) + \underline{\Phi}_2 \underline{x}_2(j) \qquad (4.9)$$

The solution in Equation (4.7) is identical to the result presented by Anderson [4] when the latter is applied to an unforced model. However, the method of specifying the data which is represented by $\underline{Z}$ and $\underline{W}$ is different. Anderson fitted the reduced order model to an open loop response of the high order model. He did not, however, specify the disturbance used to generate this response. Thus, in his presentation, the vectors which make up the rows of $\underline{Z}$ ($\underline{x}_1^T(1)$, $\underline{x}_1^T(2)$, $\underline{x}_1^T(3)$, etc.) are consecutive points along a particular open-loop response. In the work outlined here, the elements of the vectors $\underline{x}_1(j)$ and $\underline{x}_2(j)$, but not $\underline{x}_1(j+1)$, used in Equations (4.8) and (4.9), are uniformly distributed random numbers. This was done in order to overcome some of the difficulties with the trajectory fitting approach. With the use of random numbers, it is not necessary to obtain a linearly independent set of inputs to use in the calculation of the time domain response. Also, using the random numbers, data from the entire range of possible states is used. This will eliminate the bias towards a particular set of inputs which is found in the trajectory fitting approach.

4.2.2 Calculation of $\hat{\underline{\Delta}}_R$ for a Stable System

Once $\underline{\Phi}_R$ has been obtained, $\hat{\underline{\Delta}}_R$ is calculated to provide steady state agreement between the high order model and the reduced order model for step changes in $\underline{u}$. This results in the following expression

$$\hat{\underline{\Delta}}_R = (\underline{I} - \underline{\Phi}_R)\ [(\underline{I} - \underline{\Phi})^{-1}\ \hat{\underline{\Delta}}]_\ell \tag{4.10}$$

where the notation $[\underline{E}]_\ell$ represents a matrix formed by the first ℓ rows of $\underline{E}$.

4.2.3 Calculation of Both $\underline{\Phi}_R$ and $\hat{\underline{\Delta}}_R$ Using Least Squares

An alternate approach for calculating a reduced order model using least squares, is to evaluate both $\underline{\Phi}_R$ and $\hat{\underline{\Delta}}_R$ at the same time. To do this, the general approach outlined in Section 4.2.1 is followed, except that both $\underline{\Phi}_R$ and $\hat{\underline{\Delta}}_R$ are calculated by Equation (4.7) as

$$\begin{pmatrix} \underline{\Phi}_R^T \\ \hat{\underline{\Delta}}_R^T \end{pmatrix} = (\underline{Z}^T\ \underline{Z})^{-1}\ \underline{Z}^T\ \underline{W} \tag{4.11}$$

where

$$\underline{Z} = \begin{bmatrix} \underline{x}_1^T(1),\ \hat{\underline{u}}^T(1) \\ \underline{x}_1^T(2),\ \hat{\underline{u}}^T(2) \\ \vdots \\ \underline{x}_1^T(k),\ \hat{\underline{u}}^T(k) \end{bmatrix} \tag{4.12}$$

and the rows of $\underline{W}$ are calculated using

$$\underline{w}(j) = \underline{x}_1(j+1) = \underline{\Phi}_1\ \underline{x}_1(j) + \underline{\Phi}_2\ \underline{x}_2(j) + \hat{\underline{\Delta}}_1\ \hat{\underline{u}}(j) \tag{4.13}$$

Here, all the elements of $\underline{x}_1$, $\underline{x}_2$ and $\hat{\underline{u}}$ are specified as random numbers.

The resulting reduced order model, as calculated by Equation (4.11), is not guaranteed to provide steady state agreement between the high order and the reduced order models. A reduced order model with better steady state agreement can be calculated by

adding steady state points to matrices $\underline{Z}$ and $\underline{W}$, as is discussed by Anderson [1], but this will be at the expense of poorer transient agreement between the two models. A second approach to provide steady state agreement is to recalculate $\hat{\underline{\Delta}}_R$ using Equation (4.10).

4.2.4 Reduction of Systems with Integrating States

For systems with at least one integrating state (with at least one eigenvalue of $\underline{\Phi}$ equal to one), the model reduction approach outlined in Sections 4.2.1 and 4.2.2 must be revised slightly. The reduced order state vector $\underline{x}_1$ can be partitioned as

$$\underline{x}_1 = \begin{pmatrix} \underline{x}_I \\ \underline{x}_N \end{pmatrix} \tag{4.14}$$

corresponding to the integrating, $\underline{x}_I$, and non-integrating, $\underline{x}_N$, state variables of $\underline{x}_1$. Matrix $\underline{\Phi}_R$ can be similarly partitioned as

$$\underline{\Phi}_R = (\underline{\Phi}_I, \underline{\Phi}_N) \tag{4.15}$$

The partition corresponding to the integrating states can be pre-specified as

$$\underline{\Phi}_I = \begin{pmatrix} \underline{I} \\ \underline{0} \end{pmatrix} \tag{4.16}$$

The dimension of the identity matrix in Equation (4.16) is the same as the number of integrating states. The partition of $\underline{\Phi}_R$ corresponding to the non-integrating states is calculated using least squares as

$$\underline{\Phi}_N^T = (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T \underline{W} \tag{4.17}$$

In this case

$$\underline{Z} = \begin{bmatrix} \underline{x}_N^T(1) \\ \underline{x}_N^T(2) \\ \vdots \\ \underline{x}_N^T(k) \end{bmatrix} \tag{4.18}$$

and the j^{th} row of $\underline{W}$ is defined as

$$\underline{w}(j) = \underline{\Phi}_{1N}\, \underline{x}_N(j) + \underline{\Phi}_2\, \underline{x}_2(j) \tag{4.19}$$

where $\underline{\Phi}_1$ has been partitioned as

$$\underline{\Phi}_1 = (\underline{\Phi}_{1I}, \underline{\Phi}_{1N}) \tag{4.20}$$

corresponding to the partitioning of $\underline{x}_1$ in Equation (4.14). The elements of vectors $\underline{x}_N$ and $\underline{x}_2$ are specified as random numbers.

Once $\underline{\Phi}_R$ has been calculated, $\hat{\underline{\Delta}}_R$ is obtained, by columns, for agreement at large times, as

$$\hat{\underline{\Delta}}_R^i = \frac{\underline{x}_1^i(j+1) - \underline{\Phi}_R\, \underline{x}_1^i(j)}{\hat{u}_i} \tag{4.21}$$

where $\hat{\underline{\Delta}}_R^i = i^{th}$ column of $\hat{\underline{\Delta}}_R$

$\hat{u}_i = i^{th}$ element of $\hat{\underline{u}}$

$\underline{x}_1^i = \underline{x}_1$ for large values of the counter j for a step change in $\hat{u}_i$ ($\hat{u}_k = 0,\ k \neq i$)

The values of the elements of $\underline{x}_1^i(j+1)$ and $\underline{x}_1^i(j)$ corresponding to the non-integrating states are equivalent and are the steady state values of these states for a non-zero value of $\hat{u}_i$ ($\hat{u}_k = 0,\ k \neq i$).

The values of the elements of $\underline{x}_1^i(j+1)$ corresponding to the integrating states are calculated as

$$\underline{x}_1^i(j+1) = \underline{\Phi}_1 \underline{x}_1^i(j) + \underline{\Phi}_2 \underline{x}_2(j) + \hat{\underline{\Delta}}_1^i \hat{u}_i \tag{4.22}$$

where $\hat{\underline{\Delta}}_1^i$ = i^{th} column of $\hat{\underline{\Delta}}_1$ and the elements of $\underline{x}_1^i(j)$ corresponding to the integrating states are specified by random numbers and the elements of $\underline{x}_1^i(j)$ corresponding to the non-integrating states are the above mentioned steady state values for the same non-zero value of $\hat{u}_i$.

Equation (4.21) provides for steady state agreement of the non-integrating states, and agreement of the rate of change of the integrating states after the transients die out.

4.2.5 Least Squares Reduction of Continuous-Time Models

The above procedures can be applied to the reduction of continuous-time models in the form:

$$\begin{pmatrix} \dot{\underline{x}}_1 \\ \dot{\underline{x}}_2 \end{pmatrix} = \begin{pmatrix} \underline{A}_1 & \underline{A}_2 \\ \underline{A}_3 & \underline{A}_4 \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} + \begin{pmatrix} \hat{\underline{B}}_1 \\ \hat{\underline{B}}_2 \end{pmatrix} \hat{\underline{u}} \tag{4.23}$$

to give a reduced-order model in the form:

$$\dot{\underline{x}}_1 = \underline{A}_R \underline{x}_1 + \hat{\underline{B}}_R \hat{\underline{u}} \tag{4.24}$$

For the case of a stable system, with all the eigenvalues of $\underline{A}$ negative

$$\underline{A}_R^T = (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T \underline{W} \tag{4.25}$$

where $\underline{Z}$ and $\underline{W}$ are defined as in Equation (4.8) except that

$$\underline{w}^T(j) = \dot{\underline{x}}_1 = \underline{A}_1 \underline{x}_1(j) + \underline{A}_2 \underline{x}_2(j) \qquad (4.26)$$

Also, for agreement between the responses of the high order and the low order models at steady state

$$\hat{\underline{B}}_R = \underline{A}_R \left[\underline{A}^{-1} \hat{\underline{B}}\right]_\ell \qquad (4.27)$$

For the reduction of continuous-time models which contain integrating states (at least one eigenvalue of $\underline{A}$ equals zero), this procedure can be modified in a similar manner as the discrete-time procedure was modified in Section 4.2.4.

4.2.6 Other Applications of Least Squares with Random Data

The use of random data with the least squares analysis could be used whenever the data to be fitted can be generated from a mathematical relationship. For example, this would be the case if a polynomial approximation to a part of a trigonometric function was needed. This approach is described in Chapter Six, where it is used to generate incomplete state feedback control laws when the more complex, high order control law is known.

4.3 APPLICATION TO THE TENTH ORDER EVAPORATOR MODEL

The least squares reduction method using random data, as described above, has been applied to the reduction of the tenth order evaporator model in its open-loop form, as presented in Appendix A, and in a closed-loop form, with B1 controlling W1 and B2 controlling W2. The closed-loop form was obtained using

$$B1 = 0.9203\ W1$$

$$B2 = 2.768\ W2$$

These two feedback relations change the unity eigenvalues of $\underline{\Phi}$ and change W1 and W2 from integrating states to non-integrating states. This closed-loop model is also shown in Appendix A. All the reduced order models are presented in Appendix F.

4.3.1 Computational Aspects

The random numbers required for the approach of this chapter were calculated using the IBM Scientific Subroutine Package (SSP) program RANDU, as described by IBM [10,11]. The numbers calculated by RANDU are referred to as random numbers even though they are really only pseudo-random, having been calculated by a mathematical algorithm and not by a truly random process. However, since these numbers are generated systematically using a computer routine, it is possible to duplicate any sequence for purposes of comparison. The random numbers were uniformly distributed between -1 and +1, which corresponds to process values of from zero to twice the process steady state, since the variables in the model are in normalized perturbation form.

The computer storage requirements for the least squares program was reduced by using the partitioned approach to the evaluation of Equation (4.7) suggested by Anderson [3]. In this approach matrices $\underline{Z}$ and $\underline{W}$ are partitioned as

$$\underline{Z} = \begin{bmatrix} \underline{Z}_1 \\ \underline{Z}_2 \\ \underline{Z}_3 \\ \vdots \end{bmatrix} \quad \text{and} \quad \underline{W} = \begin{bmatrix} \underline{W}_1 \\ \underline{W}_2 \\ \underline{W}_3 \\ \vdots \end{bmatrix} \tag{4.28}$$

where the $\underline{Z}_i$ are square matrices of order ℓ and the $\underline{W}_i$ matrices have ℓ rows each. Equation (4.7) can now be written as

$$\underline{\Phi}_R = \left[\underline{Z}_1^T \underline{Z}_1 + \underline{Z}_2^T \underline{Z}_2 + \underline{Z}_3^T \underline{Z}_3 + \cdots\right]^{-1} \left[\underline{Z}_1^T \underline{W}_1 + \underline{Z}_2^T \underline{W}_2 + \underline{Z}_3^T \underline{W}_3 + \cdots\right] \tag{4.29}$$

This reduces the computer storage requirement by a great deal since each of the terms on the right hand side of Equation (4.29), $\underline{Z}_i^T \underline{Z}_i$ and $\underline{Z}_i^T \underline{W}_i$ can be evaluated as the data is collected. Furthermore, a $\underline{\Phi}_R$ matrix can be calculated for each $\underline{Z}_i^T \underline{Z}_i$ term which is obtained. $\underline{\Phi}_R(j)$ is now defined to include j terms in each of the sums in Equation (4.29) as

$$\underline{\Phi}_R(j) = \left[\sum_{i=1}^{j} \underline{Z}_i^T \underline{Z}_i\right]^{-1} \left[\sum_{i=1}^{j} \underline{Z}_i^T \underline{W}_i\right] \tag{4.30}$$

Now, new values of $\underline{\Phi}_R$ are calculated until $\underline{\Phi}_R(j)$ and $\underline{\Phi}_R(j-1)$ become sufficiently close as

$$\frac{\left\|\underline{\Phi}_R(j)\right\| - \left\|\underline{\Phi}_R(j-1)\right\|}{\left\|\underline{\Phi}_R(j)\right\|} \leq \zeta \tag{4.31}$$

where ζ is a convergence criterion parameter specified by the user and

$$\left\|\underline{E}\right\| = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \left|e_{ij}\right| \tag{4.32}$$

where $\underline{E}$ is any (n x m) matrix and e_{ij} is an element of $\underline{E}$. For most models calculated, a value of $\zeta = 10^{-6}$ was used. This resulted in values of k, in Equation (4.8), of between 151 and 516.

4.3.2 Evaporator Model Results

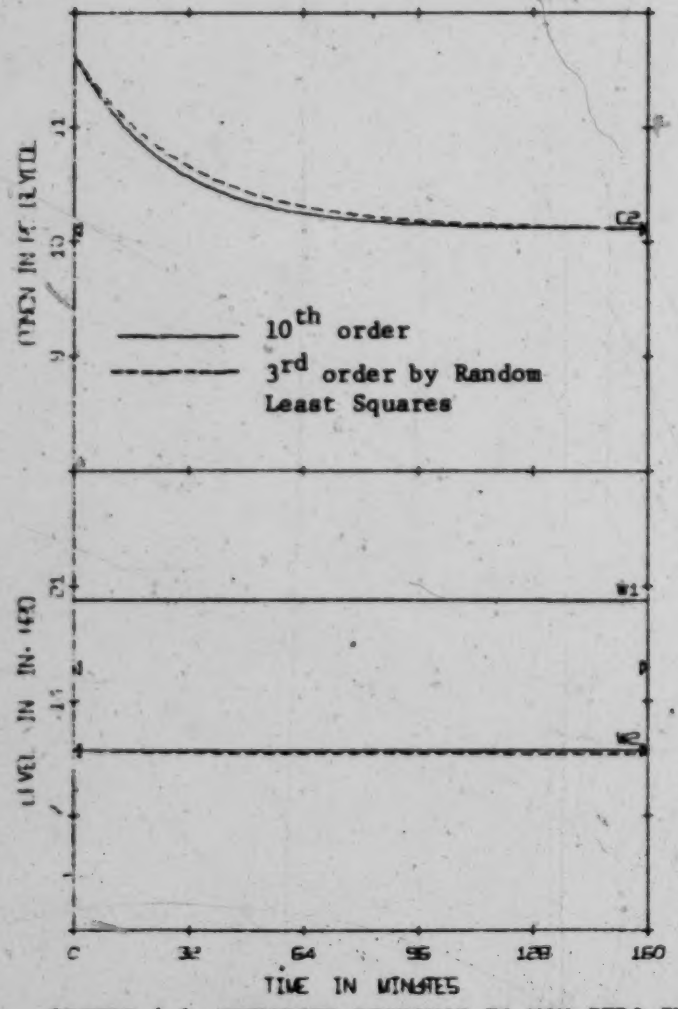
The method of least squares reduction using random data has been applied to the tenth order evaporator model. Each of the reduced order models calculated was third order with the three most important states, W1, W2 and C2 in their state vector. The results will be discussed in this section.

Unforced Response of $\underline{\Phi}_R$

In the proposed reduction method, only $\underline{\Phi}_R$ is calculated using least squares. Thus, the unforced, open-loop response of $\underline{\Phi}_R$ to a non-zero initial state should be compared to that of $\underline{\Phi}$ for the same non-zero initial state. This comparison is shown in Figure 4.1 for the initial conditions of W1(0) = +20% and C2(0) = +15%. Both holdups, W1 and W2 remain at their initial values (except for a 2% error in W2). This is expected since these are integrating states and are only slightly affected by these initial state values. The C2 responses of the two models agree very well. However, the response of C2 for the reduced order model is delayed behind that of the tenth order model. This is not observed in any of the input forcing simulations.

Different Random Number Sequences

The IBM, SSP routine, RANDU, that was used to calculate the random numbers, requires an odd integer as input data to initialize



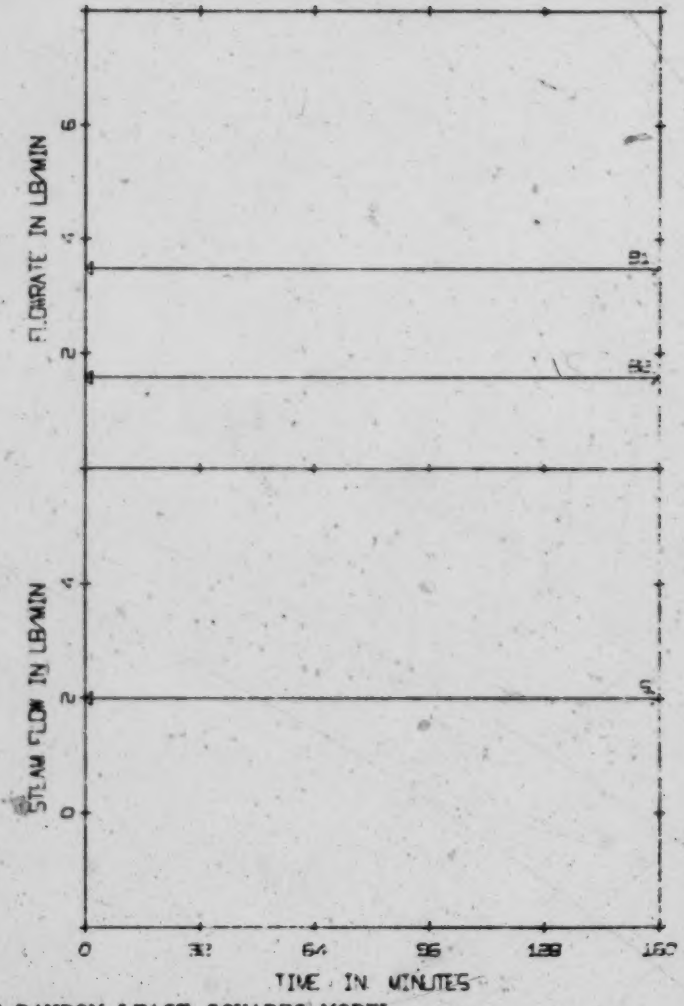
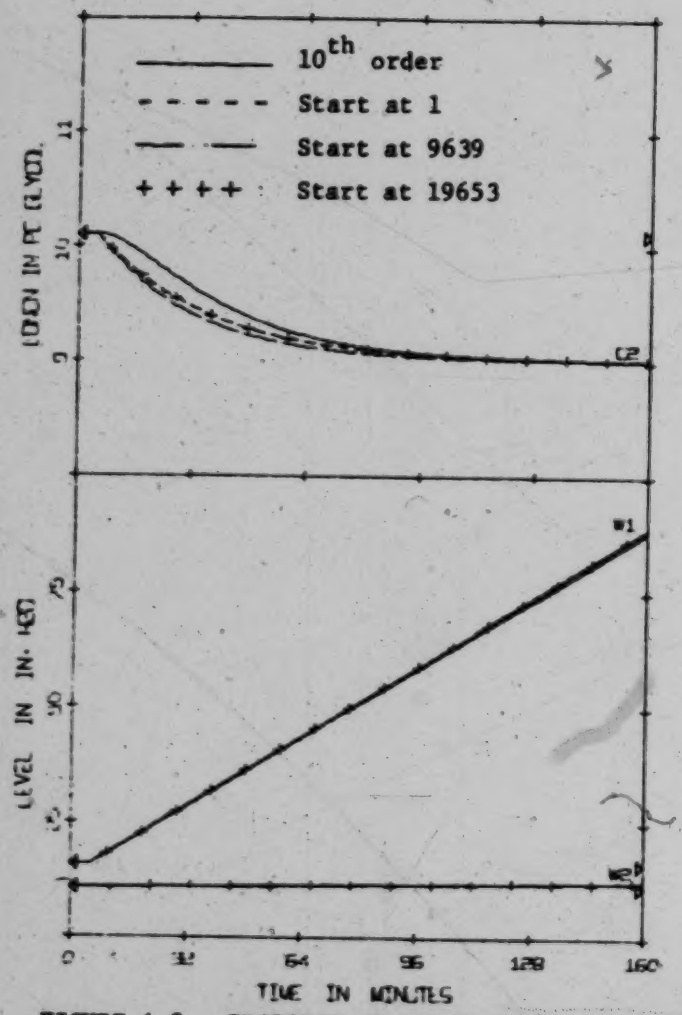


FIGURE 4.1 UNFORCED RESPONSE TO NON-ZERO INITIAL CONDITIONS OF RANDOM LEAST SQUARES MODEL (10L(2018), 3L(2119)/W1(0) = +20%, C2(0) = +15%/OL/2303, 2140)

the random number sequence. Each odd integer starts a different sequence. Thus, the response of models calculated using different number sequences must be compared. Figure 4.2 shows that the response of three different reduced order models, calculated with different number sequences, agree very well. In fact, two of these models produce identical responses to the +20% feed flow change. Furthermore, all three responses agree well with the response of the tenth order model and lead the tenth order response as would be expected. The reduced order models whose responses are shown in Figure 4.2, are open-loop models with two integrating states and are calculated using the approach outlined in Section 4.2.4.

Continuous vs. Discrete Reduction

A reduced order discrete-time model can be calculated from a high order continuous-time model along two paths, as shown in Figure 4.3. Path I obtains the discrete-time form of the high order model and reduces this using the discrete model reduction. Path II, reduces the order of the continuous-time model and then discretizes the resulting reduced order continuous-time model to get the desired reduced order discrete-time model. When a modal approach to reduction is used, Chapter Three showed that the results by Paths I and II produce an identical reduced order discrete-time model. This is not the case for least squares reduction, as is shown in Appendix B.4. Figure 4.4 demonstrates the different responses for the two reduced order discrete-time models calculated by the two paths of Figure 4.3, using the same random number sequence. However, it should be pointed out that the difference between the two reduced order responses in



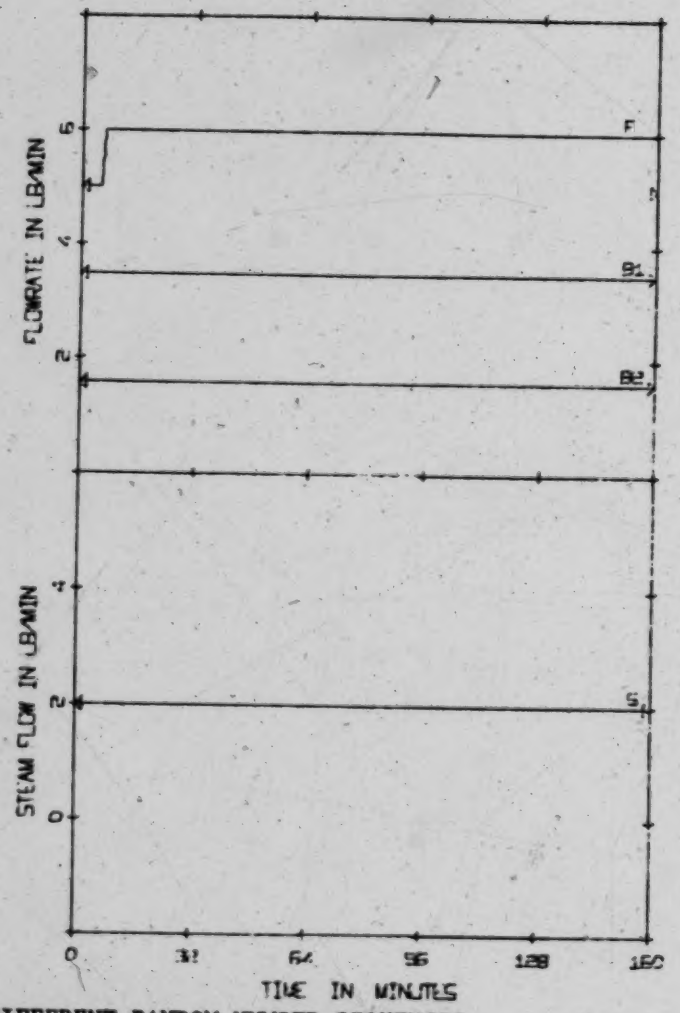


FIGURE 4.2 RESPONSE OF THIRD ORDER LEAST SQUARES MODELS FROM DIFFERENT RANDOM NUMBER SEQUENCES (10L(2018), 3L(2136), 3L(2137), 3L(2119)/D, +20%F/0L/2018, 2136, 2137, 2119)

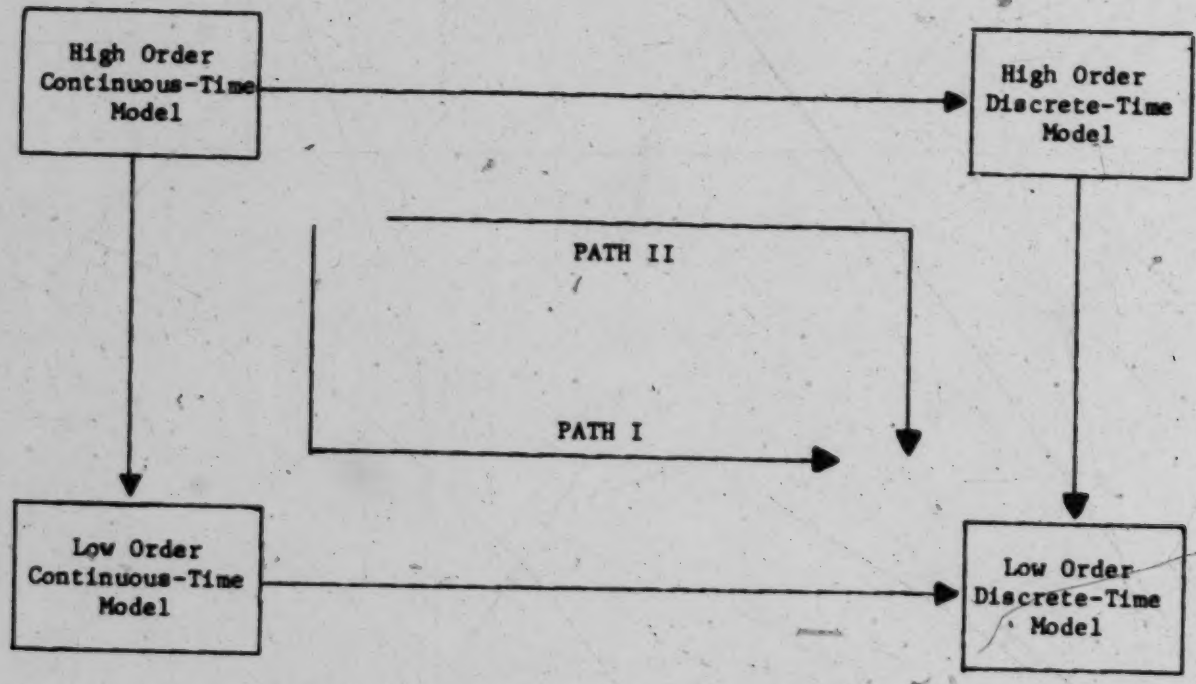
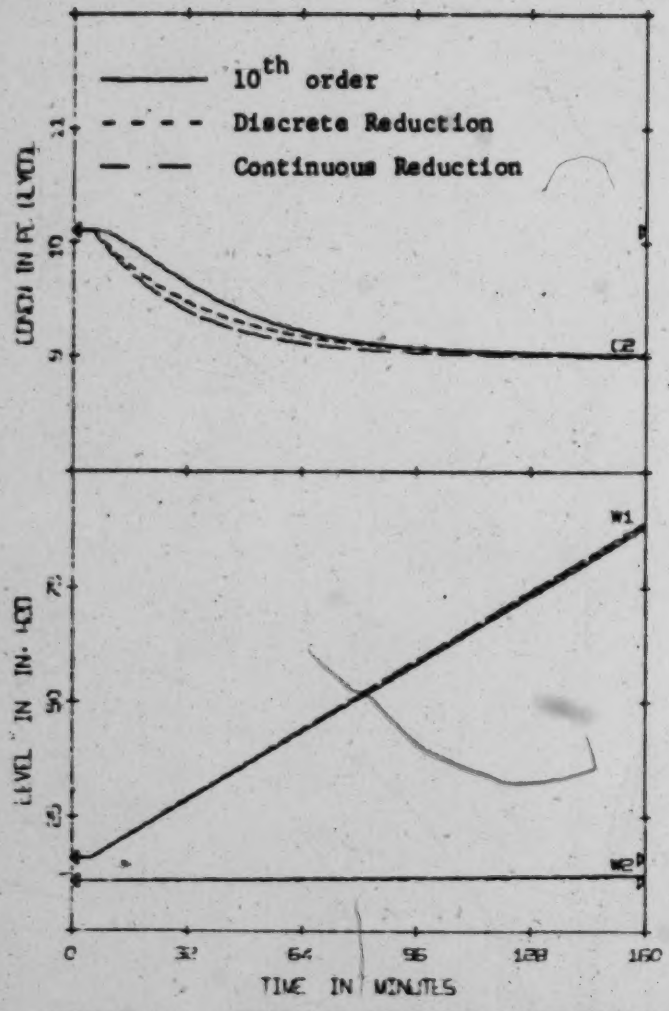


FIGURE 4.3 PATHS FOR CALCULATING A LOW ORDER DISCRETE-TIME MODEL FROM A HIGH ORDER CONTINUOUS-TIME MODEL



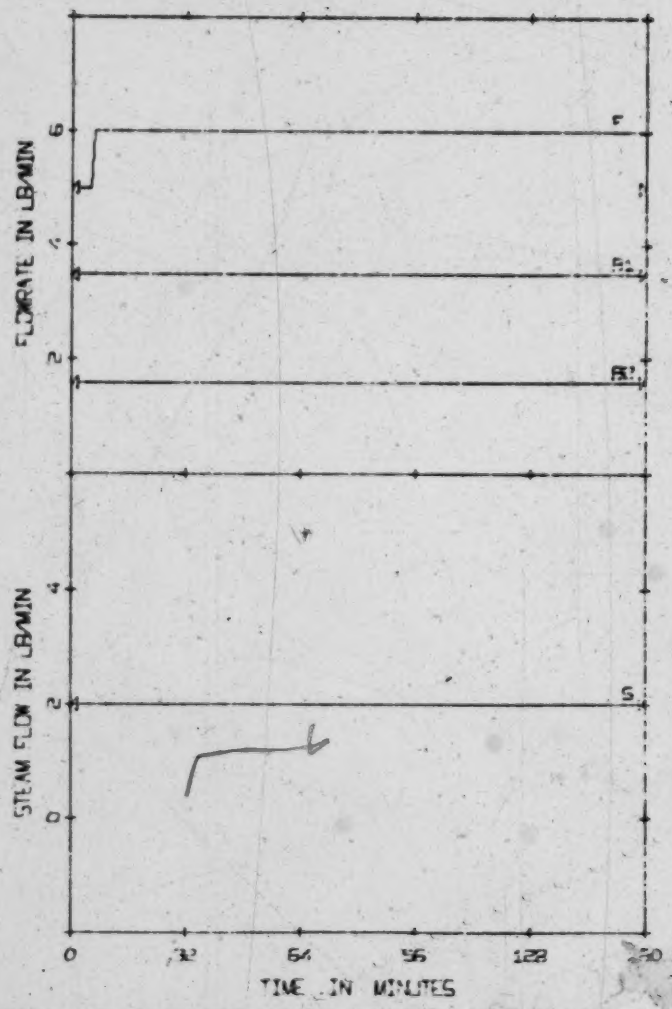


FIGURE 4.4 RESPONSE OF THIRD ORDER MODELS CALCULATED BY CONTINUOUS AND DISCRETE LEAST SQUARES REDUCTION (10L(2018), 3L(2119), 3L(2141)/D, +20%F/OL/2018, 2119, 2141)

Figure 4.4 is about the same as the difference due to different number sequences shown in Figure 4.2.

Reduction of the Closed-Loop Model

In order to test the proposed method of model reduction on a model of a different form, and also to compare it with other least squares approaches, a closed-loop evaporator model was formed as noted above. This model did not have any integrating states since the feedback control loops added changed the two unity eigenvalues of $\underline{\Phi}$ to lie between zero and one. Figure 4.5 shows that the proposed method produces a model whose response is very close to that of the tenth order model. The $\underline{\Phi}_R$ matrix for this third order model was calculated by Equation (4.7) and the $\hat{\underline{\Delta}}_R$ matrix was calculated by Equation (4.10). This model also agrees with the tenth order model at steady state. The response of the model whose $\underline{\Phi}_R$ and $\hat{\underline{\Delta}}_R$ were calculated together using least squares and random data by Equation (4.11) to (4.13) and whose $\hat{\underline{\Delta}}_R$ was not recalculated for steady state agreement, shows an expected large steady state error in Figure 4.5. The response of the model for which $\underline{\Phi}_R$ was calculated to fit an open-loop, unforced response of the tenth order model due to an initial non-zero state and $\hat{\underline{\Delta}}_R$ was calculated for steady state agreement, shows much faster dynamics than do any of the other models shown in Figure 4.5. The initial state of the trajectory, to which $\underline{\Phi}_R$ of the last model was fitted, was chosen by random selection. This may have caused the difficulty by not having all the modes of $\underline{\Phi}$ sufficiently excited.

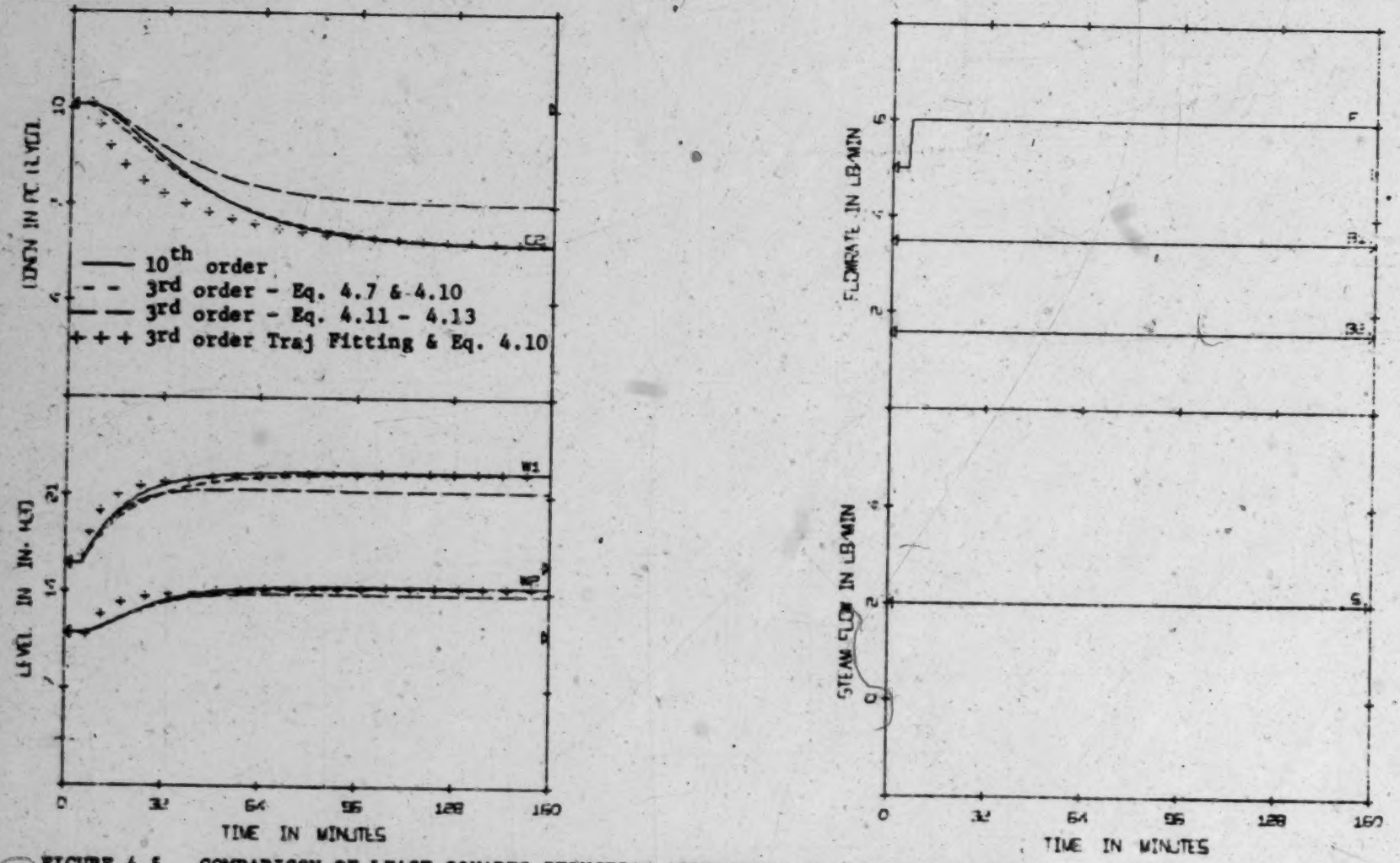


FIGURE 4.5 COMPARISON OF LEAST SQUARES REDUCTION APPROACHES ON CLOSED-LOOP MODEL (10L(2107), 3L(2113), 3L(2108), 3L(2144)/D, +20%F/OL/2022, 2113,2108, 2148)

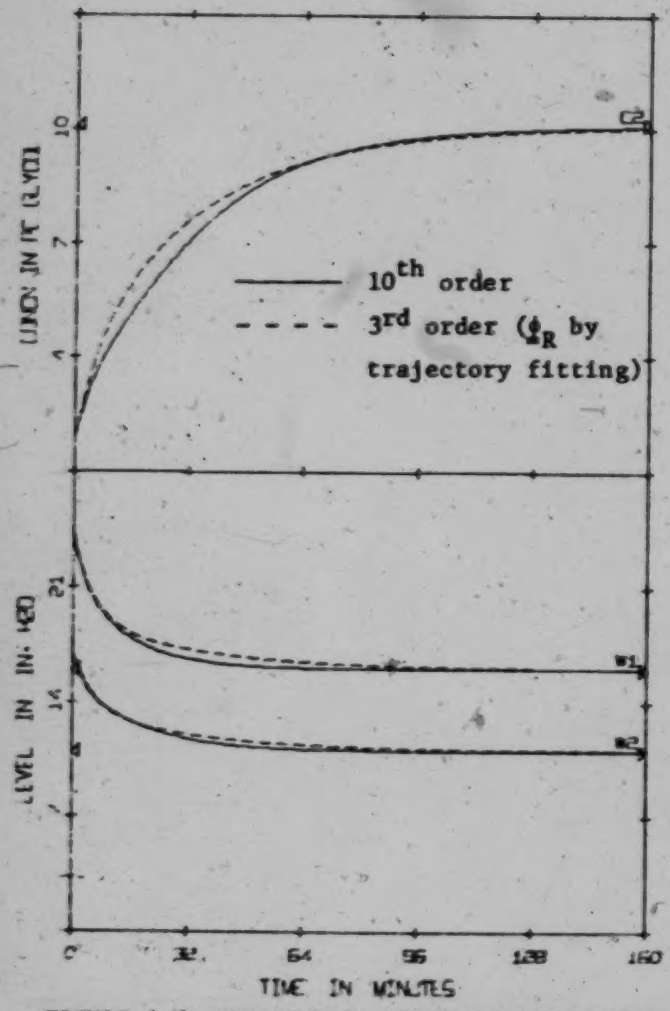
4.4 DISCUSSION

There are several advantages to the method of reduction using least squares with random data.

The first advantage is that the resulting model is not biased towards any particular set of conditions. Often, when a reduced order model is calculated, using least squares, to fit a particular response, it works well for that response, but not particularly well for any other response. An example is the model whose response is shown as the plus signs in Figure 4.5. It gave poor agreement with the high order model for this case of +20% change in feed flow. However, as Figure 4.6 shows, it agreed well with the high order model under the conditions for which it was calculated.

The second advantage of the random data approach is its ease of application. The results shown in this chapter that were calculated using this method were obtained with one computer run each. However, for the one model calculated by fitting the unforced response, several computer runs were required before it was determined how many points along the response should be considered. The convergence test of Equation (4.31) could not be used because this caused the use of too many steady state points and the corresponding response had even faster dynamics than the response shown in Figure 4.5 by the plus signs.

A third advantage of the proposed approach is in the conditioning of the matrix $(\underline{Z}^T \underline{Z})$ which must be inverted. As indicated by Ralston [16, page 233], a matrix is ill-conditioned if the inverse



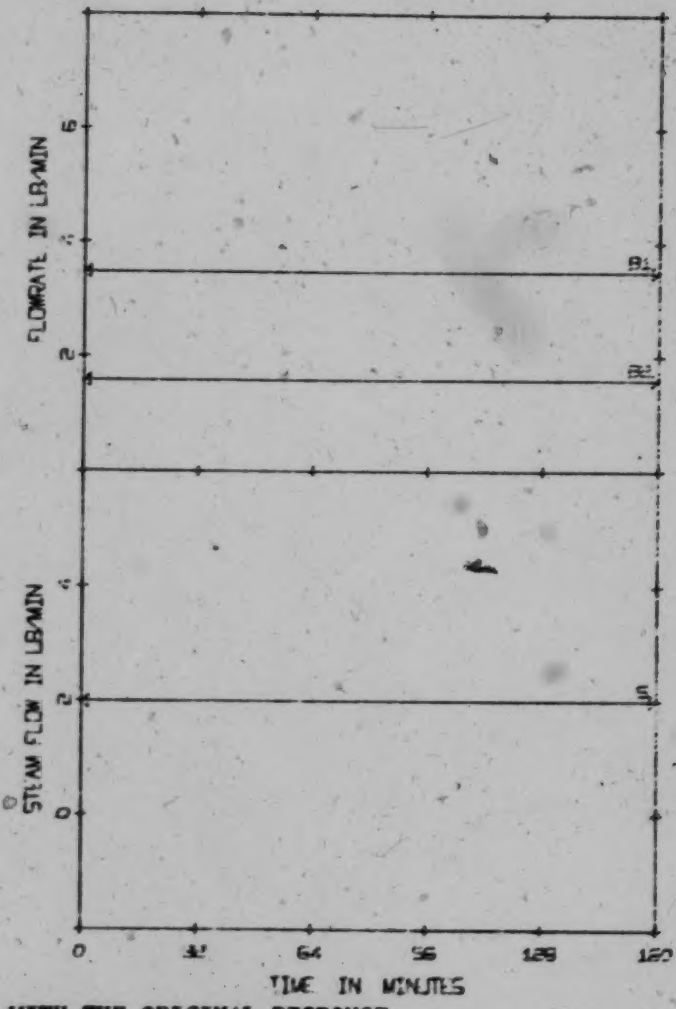


FIGURE 4.6 COMPARISON OF TRAJECTORY FITTED LEAST SQUARES MODEL WITH THE ORIGINAL RESPONSE (10L(2107), 3L(2144)/W1(0) = +39.96%, W2(0) = +39.73%, C2(0) = -81.24%/0L/2142, 2144)

of the matrix (normalized so that the largest element is one) has very large elements with respect to one. For each case run in this work, the inverse of this normalized form of $(\underline{Z}^T \underline{Z})$ was calculated. For all cases when the random data approach was used, the largest element of this inverted matrix was about 1.3, the same order of magnitude as the largest element before inversion. On the other hand, in the open-loop response fitting cases tried, the largest element of this inverted matrix was at least of order 10^4, and often much higher. Thus, for the examples used in this work, the approach using random numbers produced a much better conditioned set of equations to solve.

4.5 CONCLUSIONS

The approach to least squares model reduction calculates better reduced order models of the tenth order evaporator using random data than using data from an open-loop model trajectory. At the same time, this random data approach is easier to apply.

When a reduced order discrete-time model is calculated from a high order continuous-time model, different results are obtained using least squares depending upon whether the reduction or the discretization step is performed first.

CHAPTER FIVE

MODAL APPROACH TO CONTROL LAW REDUCTION

ABSTRACT

A method is proposed for reducing the complexity of state feedback control laws for high order systems. Using a modal approach, selected state variables are eliminated to give an incomplete state feedback control law. The proposed method for control law reduction is applied to an optimal control system for a tenth order model of a pilot scale evaporator. Simulation and experimental results show that this modal approach is practical and results in better control than methods based on reduced order models or conventional multiloop design techniques.

5.1 INTRODUCTION

The object in this investigation is to develop a systematic approach for reducing the complexity of multivariable controllers for high order systems. It is assumed that the system to be controlled can be adequately described by a linear, time-invariant state space model and that satisfactory control can be achieved using a multivariable state feedback controller which may also include integral feedback, feedforward, and setpoint control modes. Simplification of the control law is achieved by eliminating selected state variables to give an incomplete state feedback control law. The motivation for this design approach is that in most practical applications, any degradation in system performance will be offset by the practical advantages of fewer measurements, or state estimates, being required and reduced demands on the on-line computer. The simplified control law is determined, a priori, by off-line calculations.

The first step in the proposed approach is to select a control law which will give satisfactory control performance when applied to the best available model of the process. In this work, the control law is designed by applying optimal control techniques to a linearized process model. The resulting controller is a linear, time-invariant feedback control law that utilizes the entire state vector. Simplification of this control law is accomplished by eliminating selected state variables from the feedback control terms. The problem of deriving an incomplete state feedback control law from a state feedback control law will be

referred to in the study as "control law reduction", and the resulting controller will be referred to as a reduced order control law.

5.1.1 Previous Work

The design of incomplete state feedback and output feedback control systems for linear systems has been the subject of a large number of investigations in recent years. Optimal control formulations in which the control configuration is constrained to be a function of the output vector has received considerable attention by Davison [2], Kosut [6], Levine and Athans [7] and Sims and Melsa [15]. However, the calculation of these optimal control policies often requires a significant computational effort, particularly for large systems. Furthermore, the dependence of the optimal control policy on the initial state must be taken into consideration, as discussed by Kosut [6] and Levine and Athans [7].

An alternative approach considered by Anderson [1] and Nicholson [11-13] is to derive a low order state space model by model reduction techniques and then use the optimal control policy for the low order model as a suboptimal control law for the high order system. Rogers and Sworder [14] have proposed that the reduced order model and its optimal control policy be calculated simultaneously with the additional restriction that the low order control law is the best suboptimal controller for the high order system. An iterative solution of nonlinear matrix equations is required in their approach.

None of the above approaches makes direct use of an additional item of information, namely, the optimal control law for the high order

system. The strategy adopted in this investigation is to derive an incomplete state feedback law from a state feedback control law by using a modal analysis of the high order state space model to eliminate selected state variables.

5.2 CONTROL LAW REDUCTION

Consider the linear, time-invariant, discrete-time model in the form:

$$\underline{x}(j+1) = \underline{\Phi}\underline{x}(j) + \underline{\Delta}\underline{u}(j) + \underline{\Theta}\underline{d}(j) \tag{5.1}$$

$$\underline{y}(j) = \underline{C}\,\underline{x}(j) \tag{5.2}$$

where $\underline{x}$ is the n-dimensional state vector,

$\underline{u}$ is the m-dimensional control vector,

$\underline{d}$ is the q-dimensional disturbance vector,

$\underline{y}$ is the p-dimensional output vector,

$\underline{\Phi}$, $\underline{\Delta}$, $\underline{\Theta}$, $\underline{C}$ are constant coefficient matrices of appropriate dimensions,

j is a counter such that $\underline{x}(j)$ denotes $\underline{x}(t)$ at $t = jT$, where T is the discrete time interval.

It is assumed that the high order control law has the form:

$$\underline{u}(j) = \underline{K}^{FB}\underline{x}(j) + \underline{K}^{I}\sum_{i=0}^{j}\underline{y}(i) + \underline{K}^{FF}\underline{d}(j) + \underline{K}^{SP}\underline{y}^{SP} \tag{5.3}$$

where $\underline{y}^{SP}$ is the constant setpoint vector such that y_i^{SP} is the desired value of y_i, $i = 1, \ldots, p$

$\underline{K}^{FB}$, $\underline{K}^{I}$, $\underline{K}^{FF}$, $\underline{K}^{SP}$ are the proportional feedback, integral feedback, feedforward and setpoint control matrices of appropriate dimensions.

The control law of Equation (5.3) is chosen, not only because it can stabilize the system, but also because it is a practical control law which can be easily applied using an on-line digital computer as was shown by Newell [10]. The control matrices in Equation (5.3) can be determined by any applicable design technique. In this work they were calculated by dynamic programming, minimizing a quadratic performance criterion, as described by Newell [10].

The objective of this investigation is to simplify the control law by reducing the number of state variables such that

$$\underline{u}(j) = \underline{K}_R^{FB}\,\underline{x}_1(j) + \underline{K}_R^{I}\sum_{i=0}^{j}\underline{y}(i) + \underline{K}_R^{FF}\,\underline{d}(j) + \underline{K}_R^{SP}\,\underline{y}^{SP} \qquad (5.4)$$

where $\underline{x}_1$ is the ℓ-dimensional subset of $\underline{x}$ which is to be retained in the reduced order control law. Practical considerations dictate that the elements of $\underline{x}_1$ be chosen so that the system defined by Equation (5.1) can be stabilized by a control law of the form of Equation (5.4). Criteria for selecting the state variables to be retained in $\underline{x}_1$ will be discussed later.

5.2.1 Method Development

If the state vector, $\underline{x}$, is partitioned such that

$$\underline{x} = \begin{bmatrix}\underline{x}_1\\ \underline{x}_2\end{bmatrix} \qquad (5.5)$$

then the state feedback control law of Equation (5.3) can be expanded in terms of $\underline{x}_1$ and $\underline{x}_2$, to give:

$$\underline{u}(j) = \underline{\underline{K}}_1^{FB}\,\underline{x}_1(j) + \underline{\underline{K}}_2^{FB}\,\underline{x}_2(j) + \underline{\underline{K}}^{I} \sum_{i=0}^{j} \underline{y}(i) + \underline{\underline{K}}^{FF}\,\underline{d}(j) + \underline{\underline{K}}^{SP}\,\underline{y}^{SP} \tag{5.6}$$

The use of an exact expression for $\underline{x}_2$ in Equation (5.6), derived from Equations (5.1) and (5.2), would result in a complicated control law. However, the use of an approximation for $\underline{x}_2$, in terms of the other variables, would result in the desired form of the control law. Several possible approaches to this approximation exist:

1. It could be assumed that $\underline{x}_2 = \underline{0}$, which simplifies the analysis but is too extreme an assumption for most systems.
2. A second assumption is that the response of $\underline{x}_2$ is instantaneous and hence $\underline{x}_2(j)$ is an algebraic function of $\underline{x}_1(j)$, $\underline{u}(j)$, and $\underline{d}(j)$. However, it is not difficult to find systems where the response time of $\underline{x}_2(j)$ is so slow that it cannot be neglected.
3. A third approximation for $\underline{x}_2$ could be derived using a modal analysis of the high order system represented by Equation (5.1).

The modal analysis, which was selected, is analogous to that used in the reduction of continuous-time models originally proposed by Marshall [9] and as extended to discrete-time systems in Chapter Three. The basic steps are as follows.

An n x n matrix, $\underline{\underline{M}}$, exists, which transforms Equation (5.1) into its Jordan canonical form, as shown by Gantmacher [3]. This similarity transformation, which also preserves the eigenvalues of $\underline{\underline{\Phi}}$, is defined by

$$\underline{x} = \underline{M}\,\underline{z} \tag{5.7}$$

or in partitioned form

$$\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} = \begin{bmatrix} \underline{M}_1 & \underline{M}_2 \\ \underline{M}_3 & \underline{M}_4 \end{bmatrix} \begin{bmatrix} \underline{z}_1 \\ \underline{z}_2 \end{bmatrix} \tag{5.8}$$

where $\underline{z}$ is the canonical state vector and $\underline{z}_1$ and $\underline{z}_2$ have the same dimensions as $\underline{x}_1$ and $\underline{x}_2$, respectively. When Equation (5.8) is substituted into Equation (5.1), the following Jordan canonical form results (shown here in partitioned form)

$$\begin{bmatrix} \underline{z}_1\,(j+1) \\ \underline{z}_2\,(j+2) \end{bmatrix} = \begin{bmatrix} \underline{\alpha}_1 & \underline{0} \\ \underline{0} & \underline{\alpha}_2 \end{bmatrix} \begin{bmatrix} \underline{z}_1(j) \\ \underline{z}_2(j) \end{bmatrix} + \begin{bmatrix} \underline{\delta}_1 \\ \underline{\delta}_2 \end{bmatrix} \underline{u}(j) + \begin{bmatrix} \underline{\eta}_1 \\ \underline{\eta}_2 \end{bmatrix} \underline{d}(j) \tag{5.9}$$

where $\underline{\alpha}$, $\underline{\delta}$, and $\underline{\eta}$ are defined as:

$$\underline{\alpha} = \underline{M}^{-1}\,\underline{\Phi}\,\underline{M} = \begin{bmatrix} \underline{\alpha}_1 & \underline{0} \\ \underline{0} & \underline{\alpha}_2 \end{bmatrix} \tag{5.10}$$

$$\underline{\delta} = \underline{M}^{-1}\,\underline{\Delta} = \begin{bmatrix} \underline{\delta}_1 \\ \underline{\delta}_2 \end{bmatrix} \tag{5.11}$$

$$\underline{\eta} = \underline{M}^{-1}\,\underline{\Theta} = \begin{bmatrix} \underline{\eta}_1 \\ \underline{\eta}_2 \end{bmatrix} \tag{5.12}$$

Matrix $\underline{\alpha}$ is a block diagonal matrix with the eigenvalues of $\underline{\Phi}$ on its principal diagonal. It then follows directly from Equation (5.9) that each element of $\underline{z}$ is associated with a particular eigenvalue of $\underline{\Phi}$.

An expression for $\underline{x}_2$ can be calculated using Equation (5.8) and matrix $\underline{V}$ which is defined as,

$$\underline{V} = \begin{bmatrix} \underline{V}_1 & \underline{V}_2 \\ \underline{V}_3 & \underline{V}_4 \end{bmatrix} = \underline{M}^{-1} \tag{5.13}$$

to give

$$\underline{x}_2(j) = \underline{V}_4^{-1} \left[\underline{z}_2(j) - \underline{V}_3 \, \underline{x}_1(j) \right] \tag{5.14}$$

Equation (5.14) could be used to eliminate $\underline{x}_2$ from the high order control law in Equation (5.6). However, the resulting control law would then contain $\underline{z}_2$. As an approximation, the dynamics associated with $\underline{z}_2$ can be neglected and $\underline{z}_2(j)$ can be replaced by its steady state value. It will be shown later that this approximation is a reasonable one if the columns of $\underline{M}$ are arranged in an appropriate fashion. Thus, for constant input vectors, assume that $\underline{z}_2$ has reached its final steady state value; then from Equation (5.9)

$$\underline{z}_2(j) = \underline{z}_2(j+1) = (\underline{I} - \underline{\alpha}_2)^{-1} \left[\underline{\delta}_2 \, \underline{u}(j) + \underline{\eta}_2 \, \underline{d}(j) \right] \tag{5.15}$$

Equation (5.15) is then used with Equation (5.14) to give the required approximation of $\underline{x}_2$:

$$\underline{x}_2(j) = \underline{V}_4^{-1} \left[(\underline{I} - \underline{\alpha}_2)^{-1} \, \underline{\delta}_2 \, \underline{u}(j) + (\underline{I} - \underline{\alpha}_2)^{-1} \, \underline{\eta}_2 \, \underline{d}(j) - \underline{V}_3 \, \underline{x}_1(j) \right] \tag{5.16}$$

The desired reduced order control law can now be obtained by combining Equations (5.6) and (5.16) such that the coefficient matrices in Equation (5.4) are defined as:

$$\begin{aligned} \underline{K}_R^{FB} &= \underline{K}^u \, (\underline{K}_1^{FB} - \underline{K}_2^{FB} \, \underline{V}_4^{-1} \, \underline{V}_3) \\ \underline{K}_R^{I} &= \underline{K}^u \, \underline{K}^I \\ \underline{K}_R^{FF} &= \underline{K}^u \, (\underline{K}^{FF} + \underline{K}_2^{FB} \, \underline{V}_4^{-1} \, (\underline{I} - \underline{\alpha}_2)^{-1} \, \underline{\eta}_2) \end{aligned} \tag{5.17}$$

$$\underline{K}_R^{SP} = \underline{K}^u \underline{K}^{SP}$$

where

$$\underline{K}^u = (\underline{I}-\underline{K}_2^{FB} \underline{V}_4^{-1} (\underline{I}-\underline{\alpha}_2)^{-1} \underline{\delta}_2)^{-1}$$

Thus, a reduced order control law has been derived in which the feedback contribution is a function of only a selected subset of the state vector, rather than the entire state vector. It should be noted that the control system designer already has the freedom to decide what variables are included in the feedforward, integral and setpoint terms.

5.2.2 Continuous-Time Systems

A similar analysis could be used to obtain a continuous-time reduced order control law. Consider the following linear, time-invariant, continuous-time model:

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} + \underline{D}\,\underline{d} \tag{5.18}$$

$$\underline{y} = \underline{C}\,\underline{x} \tag{5.19}$$

where $\underline{x}$, $\underline{u}$, $\underline{d}$, $\underline{y}$, and $\underline{C}$ are defined earlier and $\underline{A}$, $\underline{B}$, and $\underline{D}$ are constant coefficient matrices of appropriate dimensions. This system can be stabilized by the high order control law:

$$\underline{u} = \underline{K}^{FB} \underline{x} + \underline{K}^{I} \int_0^t \underline{y}\, dt + \underline{K}^{FF} \underline{d} + \underline{K}^{SP} \underline{y}^{SP} \tag{5.20}$$

The resulting reduced, continuous-time, control system is as follows:

$$\underline{u} = \underline{K}_R^{FB} \underline{x}_1 + \underline{K}_R^{I} \int_0^t \underline{y}\, dt + \underline{K}_R^{FF} \underline{d} + \underline{K}_R^{SP} \underline{y}^{SP} \tag{5.21}$$

where

$$\underline{K}_R^{FB} = \underline{K}^u \left(\underline{K}_1^{FB} - \underline{K}_2^{FB}\, \underline{V}_4^{-1}\, \underline{V}_3\right)$$

$$\underline{K}_R^{I} = \underline{K}^u\, \underline{K}^I$$

$$\underline{K}_R^{FF} = \underline{K}^u \left(\underline{K}^{FF} - \underline{K}_2^{FB}\, \underline{V}_4^{-1}\, \underline{J}_2^{-1}\, \underline{H}_2\right)$$

$$\underline{K}_R^{SP} = \underline{K}^u\, \underline{K}^{SP}$$

and

$$\underline{K}^u = \left(\underline{I} + \underline{K}_2^{FB}\, \underline{V}_4^{-1}\, \underline{J}_2^{-1}\, \underline{G}_2\right)^{-1}$$

and $\underline{J}_2$, $\underline{G}_2$, and $\underline{H}_2$ are defined from the continuous-time canonical system as:

$$\begin{bmatrix} \dot{\underline{z}}_1 \\ \dot{\underline{z}}_2 \end{bmatrix} = \begin{bmatrix} \underline{J}_1 & \underline{0} \\ \underline{0} & \underline{J}_2 \end{bmatrix} \begin{bmatrix} \underline{z}_1 \\ \underline{z}_2 \end{bmatrix} + \begin{bmatrix} \underline{G}_1 \\ \underline{G}_2 \end{bmatrix} \underline{u} + \begin{bmatrix} \underline{H}_1 \\ \underline{H}_2 \end{bmatrix} \underline{d} \qquad (5.22)$$

$$\underline{J} = \begin{bmatrix} \underline{J}_1 & \underline{0} \\ \underline{0} & \underline{J}_2 \end{bmatrix} = \underline{V}\, \underline{A}\, \underline{M}$$

$$\underline{G} = \begin{bmatrix} \underline{G}_1 \\ \underline{G}_2 \end{bmatrix} = \underline{V}\, \underline{B}$$

$$\underline{H} = \begin{bmatrix} \underline{H}_1 \\ \underline{H}_2 \end{bmatrix} = \underline{V}\, \underline{D}$$

Matrices $\underline{M}$ and $\underline{V}$ are the same as those used for the discrete-time system, and can be calculated using either the continuous-time or the discrete-time models (See the theorem in Appendix D.1).

This continuous-time control law is subject to all the restrictions which were imposed on the discrete-time control law derived earlier.

5.2.3 Application Considerations

In applying this method, two important design decisions must be made: first, which subset of $\underline{x}$ should be included in the reduced-order control law as $\underline{x}_1$ and, secondly, which subset of the system's eigenvalues should be retained in $\underline{\alpha}_1$.

The designer must first specify which states are to be retained in the control law. This choice may require a trial and error approach, but a few general guidelines are available. For example, process variables which are to be tightly controlled or constrained must be included in the reduced order control law. Other process variables, which if not controlled would give poor system performance, must also be included in the reduced order control law. Furthermore, the state variables retained in $\underline{x}_1$ must be able to satisfactorily stabilize the high order system by means of Equation (5.4). A further consideration in the choice of which states variables to retain in $\underline{x}_1$ is whether a state variable is measurable or not. It is obvious that other considerations aside, those state variables which can be most easily or economically measured are the ones which should be retained. Unmeasurable states should be included in $\underline{x}_1$ only if they are essential for satisfactory control since a state estimation procedure, such as the Kalman filter [5] or the Luenberger observer [8] would then be required.

After the elements of $\underline{x}_1$ have been selected, the designer must assign the eigenvalues of $\underline{\Phi}$ to either $\underline{\alpha}_1$ or $\underline{\alpha}_2$ by an appropriate rearrangement of the columns of $\underline{M}$ in Equation (5.7). In general, the eigenvalues which have the most significant effect on the elements of $\underline{x}_1$ should be placed in $\underline{\alpha}_1$. The two most important factors to consider in assigning these eigenvalues, as shown by the following analysis, are the magnitude of the eigenvalues of $\underline{\Phi}$ and the magnitude of the elements of the modal matrix, $\underline{M}$. Equation (5.7) can be expanded for the i^{th} element of $\underline{x}$ as

$$x_i = m_{i1} z_1 + m_{i2} z_2 + \ldots + m_{in} z_n \qquad i = 1, 2, \ldots, n \qquad (5.23)$$

where m_{ij} is an element of $\underline{M}$ and x_i and z_i are the i^{th} element of $\underline{x}$ and $\underline{z}$ respectively. The magnitude of m_{ij} determines the effect of the j^{th} mode on the i^{th} state variable. Similarly, if $\underline{\Phi}$ has distinct eigenvalues, then the response of the k^{th} canonical state of the unforced system can be written, from Equation (5.9), as

$$z_k(j+1) = \alpha_k z_k(j) \qquad k = 1, 2, \ldots, n \qquad (5.24)$$

where α_k is an eigenvalue of $\underline{\Phi}$.

For an eigenvalue close to the origin, Equation (5.24) indicates that the corresponding mode decays rapidly and hence its effect on a particular element, x_i, of the state vector, would be short-lived. Thus, Equations (5.23) and (5.24) imply that the k^{th} mode will most significantly affect a state variable, x_i, when m_{ik} is large and α_k is also large. In this case, α_k should be included in $\underline{\alpha}_1$. In other situations, m_{ik} will be very small and thus z_k will have a negligible effect on x_i. In other situations, an eigenvalue of $\underline{\Phi}$

will be very small in magnitude and will have only a short-lived effect on x_i, reaching its steady state value quickly. In these two situations, the eigenvalues should be incorporated into $\underline{\alpha}_2$. However, in other cases, the decision may be less obvious since a large value of m_{ik} may correspond to a small value of α_k, or vice versa. After the eigenvalues have been partitioned into $\underline{\alpha}_1$ and $\underline{\alpha}_2$, the columns of $\underline{M}$ must be rearranged accordingly.

If the columns of $\underline{M}$ are arranged as above so that the modes which most affect $\underline{x}_1$ are in $\underline{z}_1$, then the modes which least affect $\underline{x}_1$ will be in $\underline{z}_2$. This means either that the m_{ij} values relating $\underline{z}_2$ to $\underline{x}_1$ are small, or that the eigenvalues in $\underline{\alpha}_2$ are small. Thus, the steady state assumption for $\underline{z}_2$ will not significantly affect $\underline{x}_1$, since the canonical states in $\underline{z}_2$ either respond faster than those in $\underline{z}_1$ or do not affect $\underline{x}_1$ due to the small values of their m_{ij} elements. Thus, the magnitude of the eigenvalues in $\underline{\alpha}_2$, and the associated m_{ij} values, give the designer a qualitative feel for the accuracy of the reduced order system. Iterative trial and error solutions to the design problem, with or without closed-loop simulations or experimental runs may be required to obtain a satisfactory reduced order control law.

One additional restriction must be imposed for systems with repeated eigenvalues which do not have linearly independent eigenvectors. The block diagonal matrices, $\underline{\alpha}_1$ and $\underline{\alpha}_2$, must be assigned so that $\underline{z}_1$ and $\underline{z}_2$ are non-interacting as was assumed in partitioning Equation (5.9). This will always be the case for systems with n linearly independent eigenvectors, since then $\underline{\alpha}$ is a diagonal matrix. However, for systems with less than n linearly independent eigenvectors, the part-

itioning of the original system must be such that it does not split a Jordan block in $\underline{\alpha}$, so that $\underline{z}_1$ and $\underline{z}_2$ will remain non-interacting. This might necessitate increasing or decreasing the size of vector $\underline{x}_1$ slightly. However, if the original system cannot be satisfactorily partitioned to keep $\underline{z}_1$ and $\underline{z}_2$ non-interacting, then the parameters in the original model could be changed by a small amount so that the eigenvectors would be linearly independent.

5.3 APPLICATION TO THE CONTROL OF AN EVAPORATOR

The proposed method was evaluated by comparing simulated and experimental response data for a pilot plant evaporator which is described in Appendix A. The simulated responses were generated from a 10^{th} order state space model of the evaporator which was derived by Newell [10]. The following control laws were compared:

(a) a third order control law designed by the proposed method using a tenth order optimal control law as the starting point.

(b) a third order control law which is optimal for a third order state space model and a quadratic performance index. The third order model was derived from the tenth order model using a model reduction technique based on the modal analysis of Marshall as presented in Chapter Three.

(c) a control law which is optimal for the tenth order model and a quadratic performance index.

(d) a conventional multiloop feedback control scheme using controller constants which were determined experimentally by Jacobson [4].

The first step in the application of the proposed method is to determine which states to include in $\underline{x}_1$. An analysis of the evap-

orator showed that three state variables, product concentration, C2, and the two holdups, W1 and W2 must be retained in $\underline{x}_1$. C2 must be retained since regulation of C2 is specified to be the primary control objective. W1 and W2 must be retained since these are integrating states (corresponding to eigenvalues of $\underline{\underline{\Phi}}$ of one) which would tend to exceed physical operating limits if they were not controlled. The set-point and integral control modes were also chosen to be a function of these three state variables, making $\underline{y} = \underline{x}_1$.

The second step in the proposed method is to choose which eigenvalues to retain in $\underline{\underline{\alpha}}_1$ by an appropriate arrangement of the columns of $\underline{\underline{M}}$ ($\underline{\underline{M}}$ is shown for the evaporator model in Appendix A). If the previous choice of $\underline{x}_1 = [W1, W2, C2]^T$ is made, then the three largest eigenvalues should be retained. This follows from an inspection of matrix $\underline{\underline{M}}$ since the three slowest modes have the most effect on the elements of $\underline{x}_1$.

Once the state vector and eigenvalues have been partitioned and the columns of $\underline{\underline{M}}$ have been arranged, the high order control law, matrix $\underline{\underline{V}}$, and the canonical state equation are partitioned as shown in Equations (5.6), (5.13), and (5.9), respectively. Then, the reduced order control matrices can be calculated using Equation (5.17).

The third order control laws which were designed by the proposed method are presented in Tables 5.1 and 5.2. Table 5.3 summarizes the values of the quadratic performance index, J,

$$J = \sum_{j=1}^{N} \left\{ (\underline{x}(j) - \underline{x}^{SP})^T \underline{\underline{Q}} (\underline{x}(j) - \underline{x}^{SP}) + \underline{u}^T(j-1) \underline{\underline{R}} \underline{u}(j-1) \right\} \qquad (5.25)$$

TABLE 5.1

PROPORTIONAL, FEEDFORWARD AND SETPOINT CONTROL

MATRICES DESIGNED BY CONTROL LAW REDUCTION

$$\underline{K}_R^{FB} = \begin{bmatrix} 2.467 & 0.02163 & -4.705 \\ 4.288 & -1.340 & 8.885 \\ 4.128 & 9.760 & 9.528 \end{bmatrix}$$

$$\underline{K}_R^{FF} = \begin{bmatrix} 1.238 & -0.5639 & -0.4138 \\ 0.9815 & 0.2177 & -0.001227 \\ 0.9978 & 0.9877 & -0.001398 \end{bmatrix}$$

$$\underline{K}_R^{SP} = \begin{bmatrix} -2.468 & -0.02119 & 5.276 \\ -4.289 & 1.340 & -9.100 \\ -4.128 & -9.763 & -10.52 \end{bmatrix}$$

TABLE 5.2

PROPORTIONAL PLUS INTEGRAL CONTROL

MATRICES DESIGNED BY CONTROL LAW REDUCTION

$$\underline{K}_R^{FB} = \begin{bmatrix} 2.977 & 0.09369 & -5.408 \\ 5.066 & -1.246 & 8.100 \\ 5.686 & 12.29 & 11.58 \end{bmatrix}$$

$$\underline{K}_R^{I} = \begin{bmatrix} 0.4117 & 0.02088 & -0.4861 \\ 0.8398 & -0.2887 & 0.8159 \\ 0.8559 & 1.945 & 1.068 \end{bmatrix}$$

TABLE 5.3

CONTROL LAW COMPARISON

Type of Control Law	Forcing function or Initial Conditions	Performance Index, J			Transient Response shown in Fig. No.
		Optimal controller for 10th order system	Reduced Control law (by proposed method)	Optimal Control law for 3rd order model	
Proportional feedback	all x_i (0)=0 except W1(0)=0.2, C2(0)=0.15	3.779	3.817	4.102	5.1
Proportional plus integral feedback	all x_i (0)=0 except W1(0)=0.2 C2(0)=0.15	4.657	4.789	4.969	5.2
Proportional feedback plus feedforward	+20% step in F	1.018	1.226	1.223	5.3
Proportional feedback plus setpoint	+7% setpoint change in C2	0.9477	1.198	1.547	5.4

obtained by controlling the tenth order model with three of the control laws mentioned above (controllers (a), (b) and (c)). The optimal tenth order control law is shown in Appendix A and was calculated using $\underline{Q}$ = diag (0,0,10,0,0,0,10,100,0,0) and $\underline{R}$ = diag (0.05,0.05,0.05). The optimal control matrices for the third order model used the same $\underline{R}$, with $\underline{Q}$ = diag (10,10,100) and are shown in Tables 5.4 and 5.5. The simulated runs presented in Table 5.3 are:

(a) Proportional feedback control with a non-zero initial condition (i.e., $\underline{x}(0) \neq \underline{0}$),

(b) Proportional-integral feedback control with a non-zero initial condition,

(c) Proportional feedback - feedforward control with a +20% feed flow disturbance (starting from $\underline{x}(0) = \underline{0}$),

(d) Proportional feedback-setpoint control with a +7% setpoint change in C2.

The results summarized in Table 5.3 show that the third order control law calculated by the proposed method was better, as indicated by the lower value of J, than the third order controller calculated from the reduced order model. The only exception to this is for the case of proportional feedback-feedforward control where the two third order control laws give almost identical values, differing by only 0.25%. In each case, the high order optimal control law gave the lowest value of J, as would be expected.

The transient responses for proportional feedback control in Figure 5.1 demonstrate that the reduced order control law calculated by the proposed method gives better control than the control law

TABLE 5.4

PROPORTIONAL, FEEDFORWARD AND SETPOINT CONTROL

MATRICES DESIGNED FROM REDUCED ORDER MODEL

$$\underline{K}_R^{FB} = \begin{bmatrix} 4.904 & -0.4013 & -11.92 \\ 5.784 & -1.600 & 4.425 \\ 4.093 & 9.685 & 9.357 \end{bmatrix}$$

$$\underline{K}_R^{FF} = \begin{bmatrix} 1.238 & -0.5583 & -0.4128 \\ 0.9832 & 0.2231 & -0.00054 \\ 0.9983 & 0.9937 & -0.00122 \end{bmatrix}$$

$$\underline{K}_R^{SP} = \begin{bmatrix} -4.904 & 0.4017 & 12.47 \\ -5.784 & 1.599 & -4.648 \\ -4.094 & -9.686 & -10.35 \end{bmatrix}$$

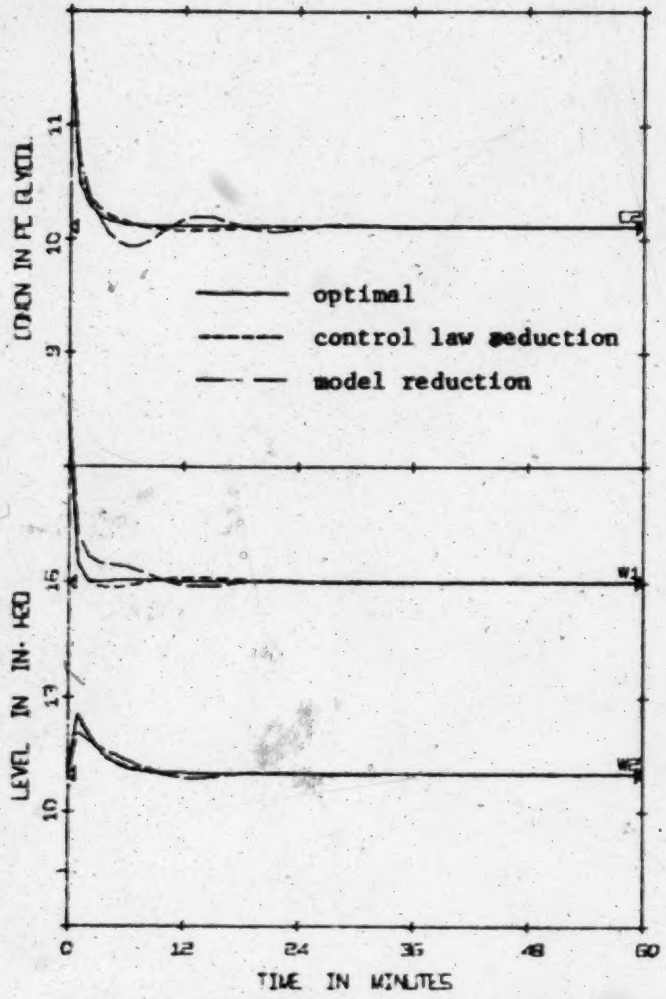
TABLE 5.5

PROPORTIONAL PLUS INTEGRAL CONTROL

MATRICES DESIGNED FROM REDUCED ORDER MODEL

$$\underline{K}_R^{FB} = \begin{bmatrix} 5.49 & -0.1903 & -12.00 \\ 6.429 & -1.386 & 4.487 \\ 5.519 & 12.26 & 11.81 \end{bmatrix}$$

$$\underline{K}_R^{I} = \begin{bmatrix} 0.9893 & -0.05066 & -1.175 \\ 1.156 & -0.3255 & 0.4373 \\ 0.8254 & 1.935 & 1.090 \end{bmatrix}$$



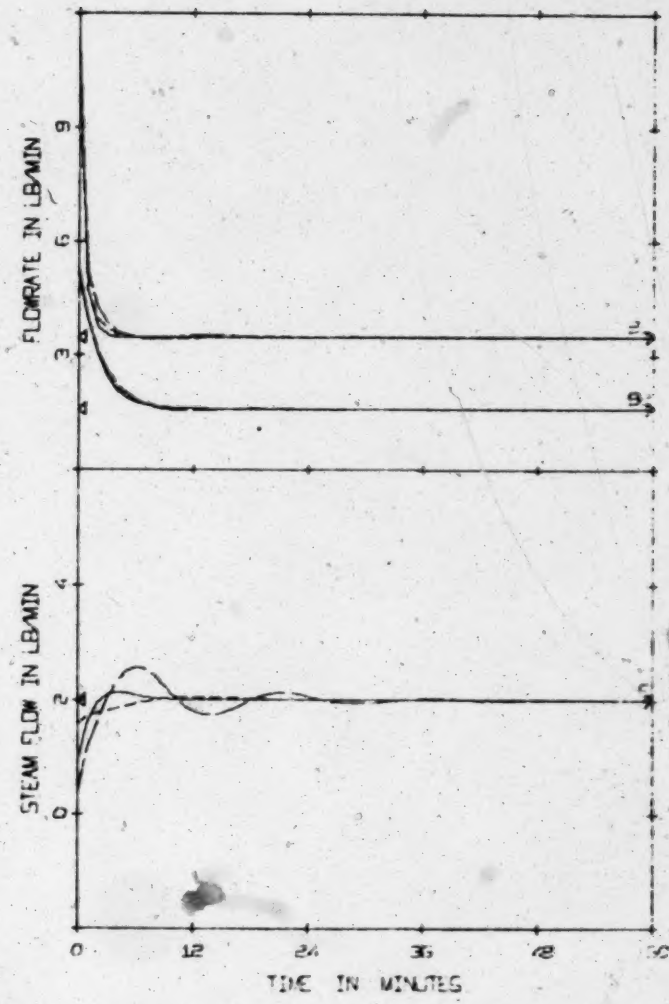


FIGURE 5.1 SIMULATED COMPARISON OF PROPORTIONAL FEEDBACK CONTROL
(10(2018)/W1(0) = +20%, C2(0) = +15%/FB/100PT, 3RED10, 30PT/3049A, 3049B, 3049C)

designed from the third order model. This is especially evident for the primary controlled variable C2, where a slight oscillation develops when the control law designed from the reduced order model is used. Figure 5.2 presents a similar comparison for proportional plus integral control law with the same initial state as was shown in Figure 5.1. Again, the proposed method is the better low order controller but the responses are more oscillatory than for proportional feedback control, as was shown in Figure 5.1.

Figure 5.3 and the performance indices in Table 5.3 indicate that the two third order feedback, feedforward control laws are quite similar. All three control laws in Figure 5.3 corrected for the +20% feed flow change with no steady state error. In Figure 5.4 the closed-loop responses of three feedback-setpoint controllers are compared for a +7% setpoint change in C2. Again, the control law designed from the third order model results in a more oscillatory response than the control law designed by the proposed method. All three responses attained the desired steady state.

In general, the effectiveness of the reduced order controller matrices, $\underline{K}_R^{FF}$, $\underline{K}_R^{SP}$ and $\underline{K}_R^{I}$, depends upon the $\underline{K}_R^{FB}$ with which they are used. This point was investigated for a proportional plus setpoint control law where $\underline{K}_R^{FB}$ was calculated from the reduced order model and used with the $\underline{K}^{SP}$ matrix which was optimal for the high order model. For the same +7% setpoint in C2, this approach gave J = 37.89, This large value was due mainly to the resulting steady state C2 value of +11.5% instead of the desired value of +7%.

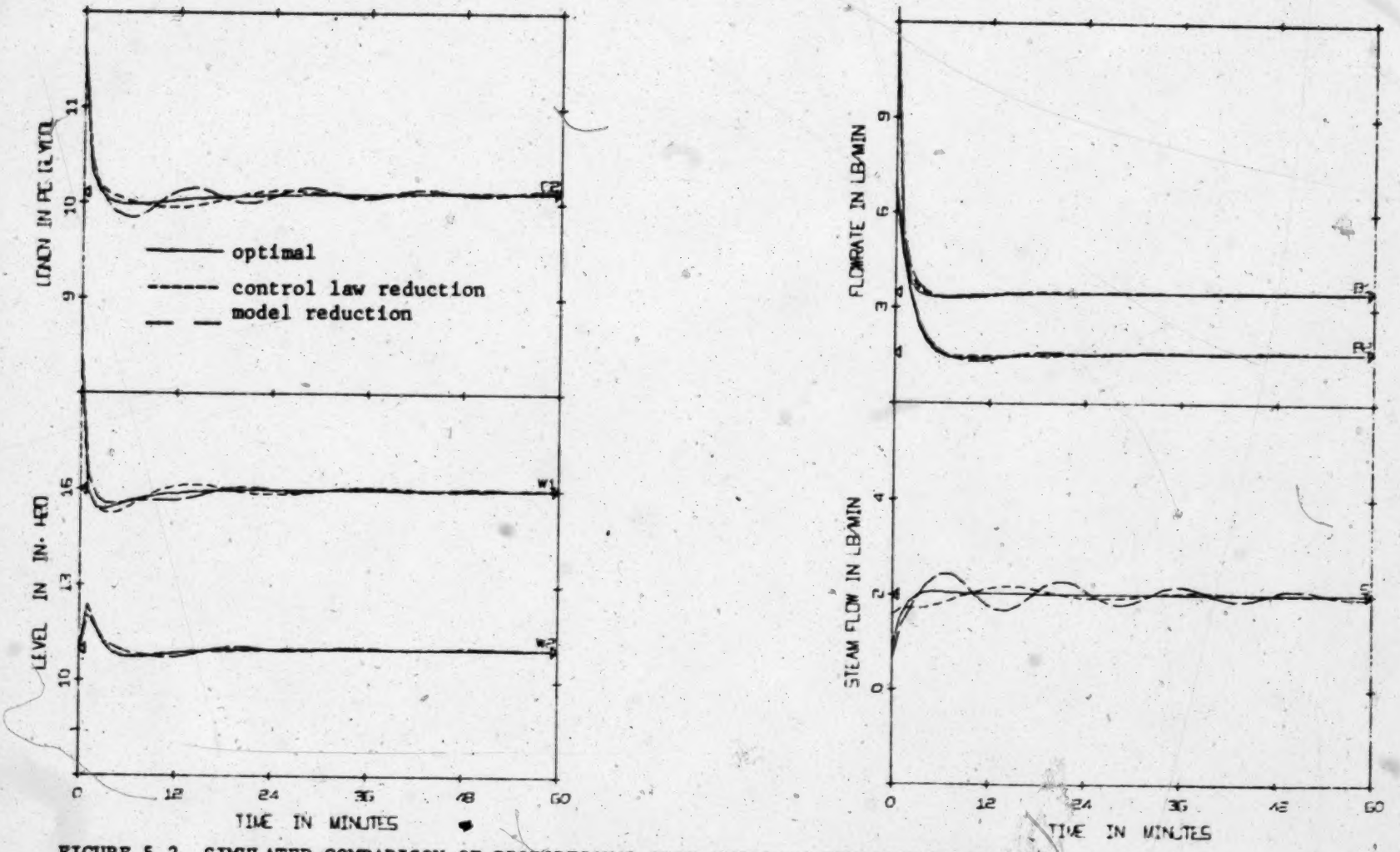


FIGURE 5.2 SIMULATED COMPARISON OF PROPORTIONAL PLUS INTEGRAL FEEDBACK CONTROL (10L(2018)/W1(0) = +20%, C2(0) = +15%/FB + I/100PT, 3RED10, 30PT/3051A, 3051B, 3051C)

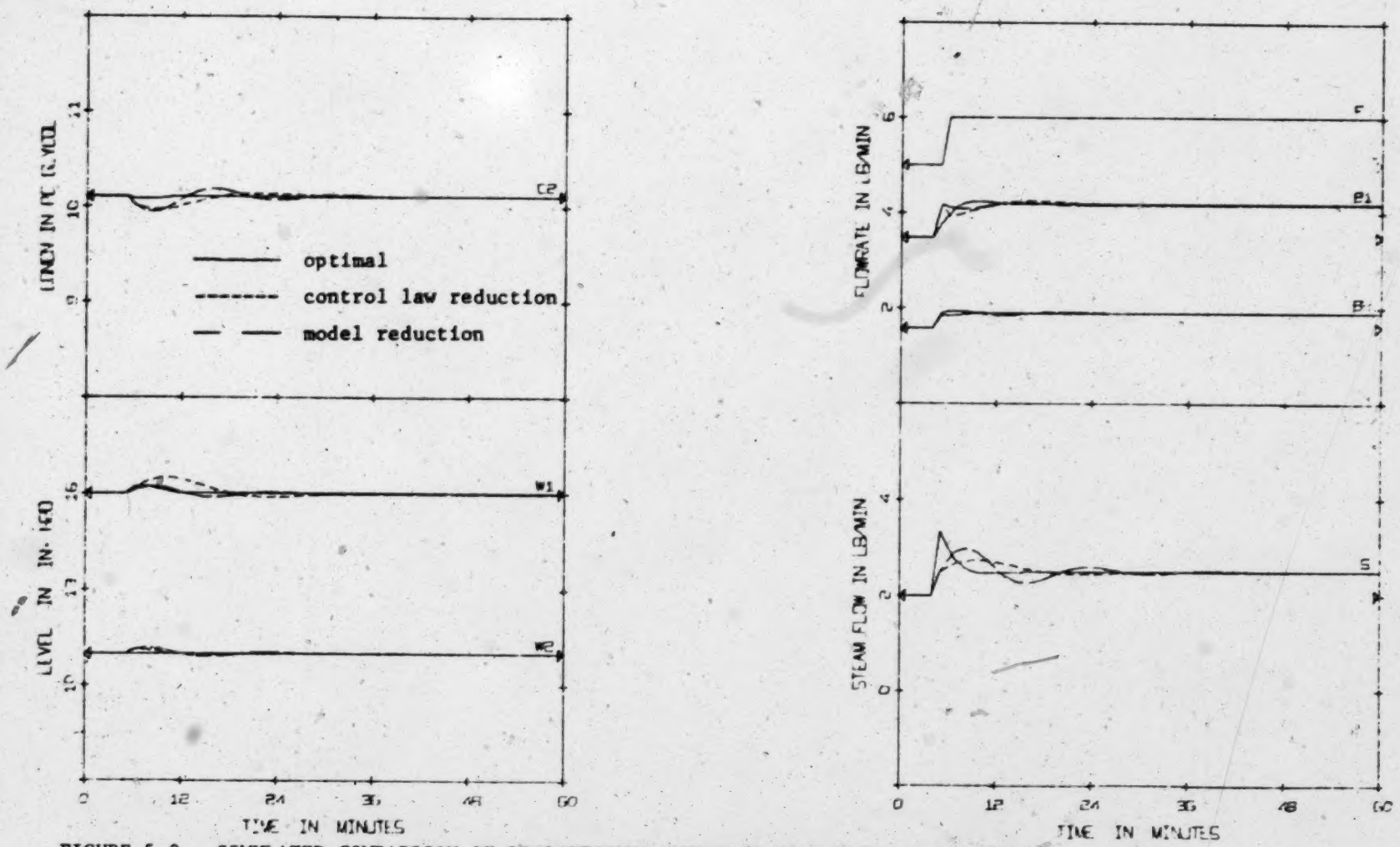


FIGURE 5.3 SIMULATED COMPARISON OF PROPORTIONAL FEEDBACK PLUS FEEDFORWARD CONTROL (10L(2018)/D, +20%F/FB + FF/100PT, 3RED10, 30PT/3016, 3017, 3019)

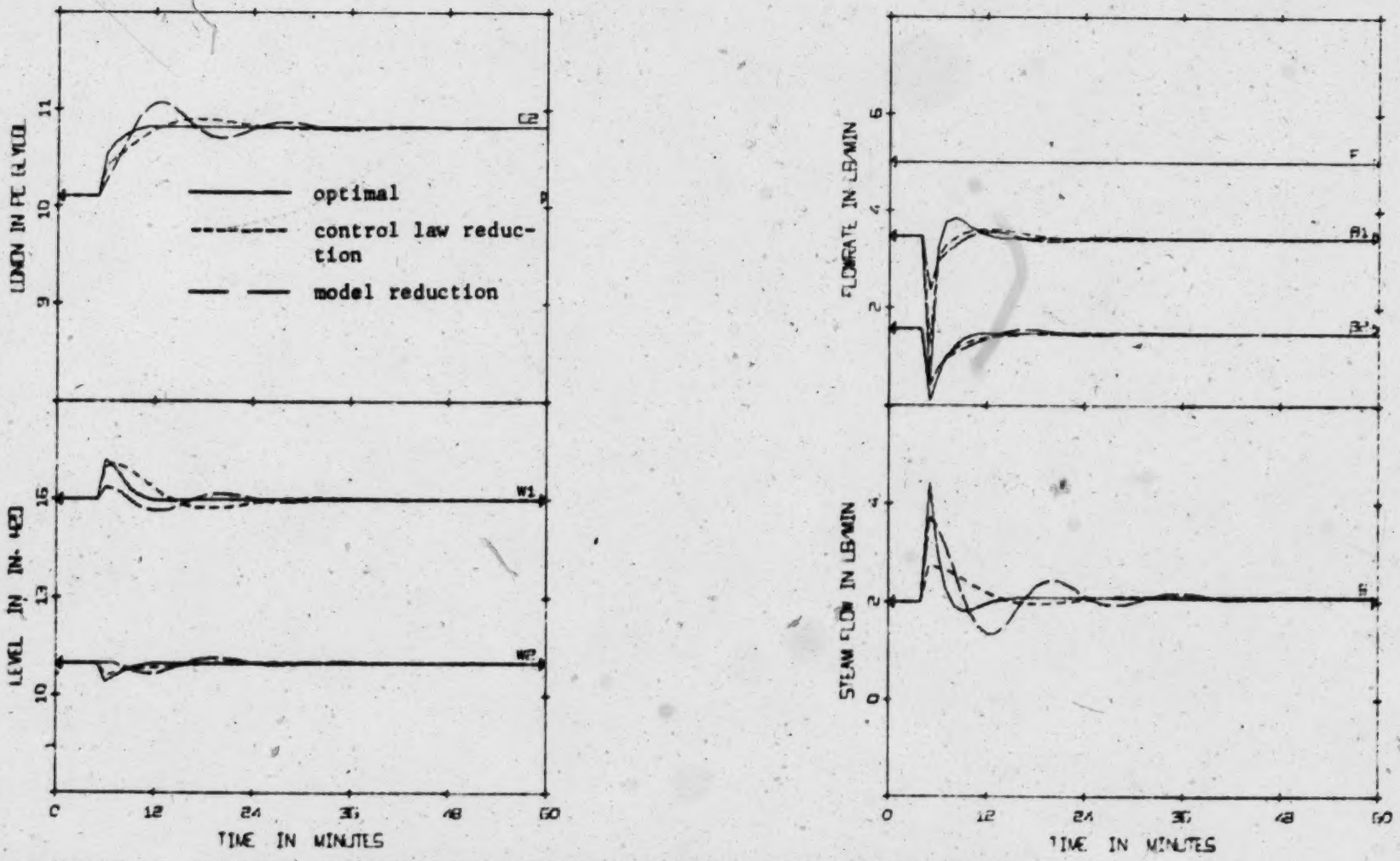


FIGURE 5.4 SIMULATED COMPARISON OF PROPORTIONAL FEEDBACK PLUS SETPOINT CONTROL (10L(2018)/SP, +7%C2/FB + SP/100PT, 3RED10, 30PT/3043A, 3044, 3043B)

In process control applications, feedback control laws are primarily used to correct for unmeasured disturbances which enter the system. Thus, it was of interest to examine the effectiveness of a controller designed by the proposed method in such situations. Figure 5.5 shows the response of the high order evaporator model when upset by a +20% feedflow disturbance and controlled by the optimal tenth order proportional feedback controller or by the third order proportional feedback control law designed by the proposed method. The third order control law results in satisfactory control but is, of course, not as good as the optimal control law.

The above examples demonstrate that the reduced order control laws designed by the proposed method give excellent results when applied to the control of the high order model. The experimental response shown in Figure 5.6 indicates that the resulting control law also performs well when controlling the actual pilot plant evaporator. The proportional control scheme results in satisfactory control of W1, W2, and C2 after two 20% step changes in feed flow. (The proportional feedback control matrix for Figure 5.6 is shown in Table 5.1). This experimental response represents a significant improvement over previously reported results for a conventional multiloop controller [4]. A comparison of control by the proposed method with control by this multiloop control scheme is presented in Chapter Six, for proportional plus integral control.

In the development of this new approach for designing a reduced order control law, it appeared that a possible application might be for

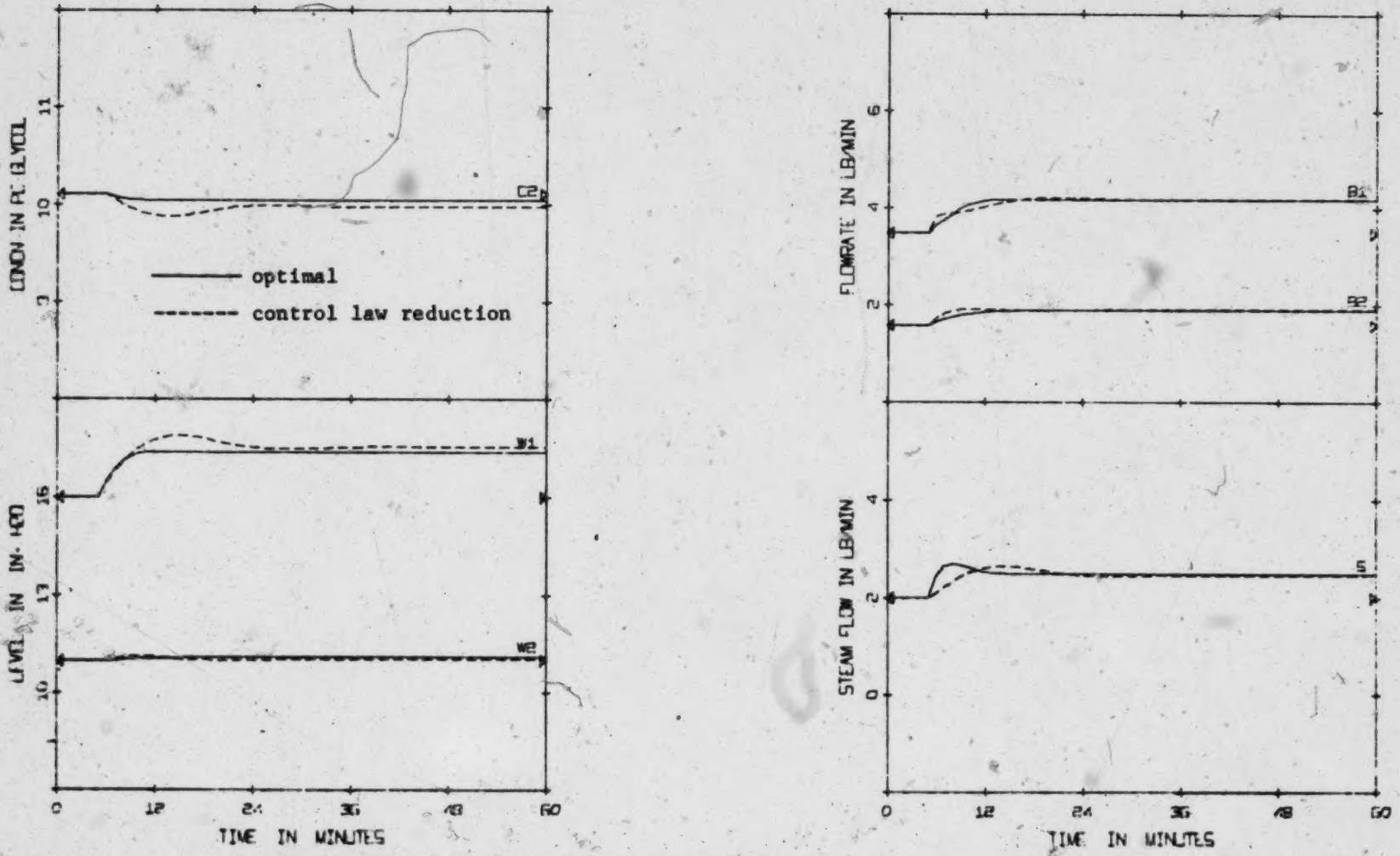


FIGURE 5.5 SIMULATED COMPARISON OF PROPORTIONAL FEEDBACK CONTROL OF A FEED DISTURBANCE (10L(20:8)/D, +20%F/FB/100PT, 3RED10/3040A, 3040B)

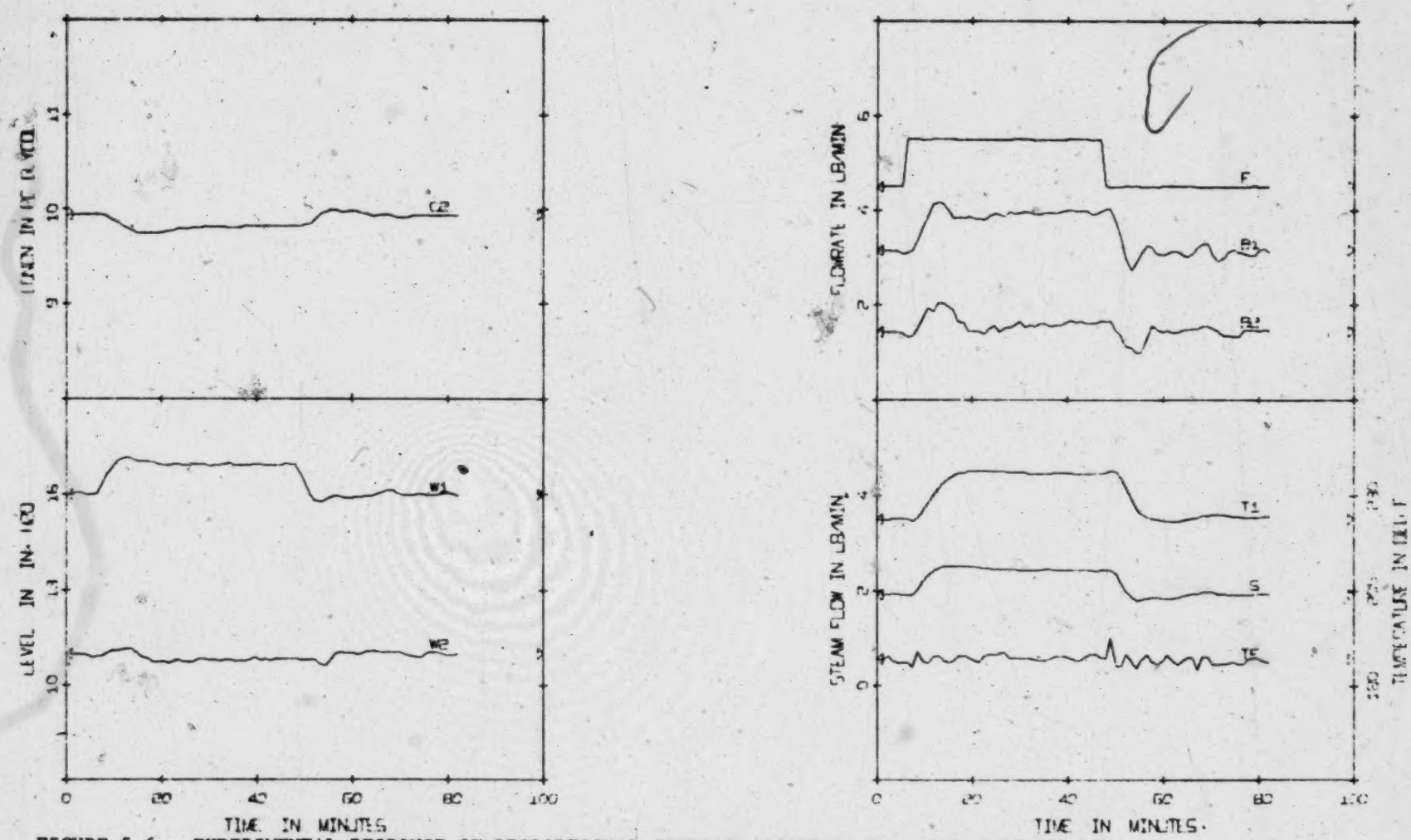


FIGURE 5.6 EXPERIMENTAL RESPONSE OF PROPORTIONAL FEEDBACK CONTROL OF A FEED DISTURBANCE (PP/D, 20%F/FB/3RED10/FB19)

"implied" control. Suppose, for example, there is a key state variable, x_k, which cannot be measured, but must be controlled. It appeared that, if the optimal, high order control law for the system was designed with heavy weighting on x_k, then the methods of this chapter could be applied to obtain a reduced order control law which was not a function of x_k, but which would still control it satisfactorily. This strategy was attempted in the evaporator simulation studies. The variable to be controlled was taken as C2 and several feedback control laws were tried, all without success. It is possible that, with further study into the structure, controllability and/or observability of the system, a satisfactory "implied" control scheme could be devised. However, this was not pursued as part of this study.

5.4 CONCLUSIONS

A method has been proposed for deriving reduced order control laws which eliminate selected state variables from high order control laws. In effect, an incomplete state feedback control law is derived from a state feedback control law using a modal analysis of the high order state space model. Both the high order and low order control laws may contain feedforward, integral, and setpoint modes. The feasibility of the proposed method has been demonstrated in simulation and experimental studies involving a pilot scale evaporator. When applied to the evaporator model, the low order control laws designed using the proposed method gave better results than controllers designed from reduced order models which had been obtained using model reduction techniques, as indicated by the smaller value of the performance index.

CHAPTER SIX

MODEL REDUCTION AND REDUCED ORDER CONTROL OF A PILOT PLANT EVAPORATOR

ABSTRACT

The literature dealing with the structure of the field of reduced order control law design is reviewed. The most promising methods were applied to the experimental control of a pilot plant evaporator. The control law which gave the best overall control, of the methods tried, was designed by the elimination of selected state variables from the feedback portion of the high order control law for the process using a modal analysis. The subject of model reduction is briefly treated. It is shown that most of the modal methods of model reduction produce one of two basic results or two modifications to one of these basic results.

6.1 INTRODUCTION

This chapter considers, in more detail, the general problems of model reduction and reduced order control law design which have been discussed for specific topics in Chapters Three, Four, and Five.

The models of some processes are of such high order, and their high order multivariable control laws so complex, that they are impractical for use in simulation and control. The simplification of process models, and the design of control laws which require only a small subset of the states of the model have been the topic of many investigations. These simplified models and control laws can then be used to study and control the process. This chapter summarizes the work done in these two general fields as well as applying some of the methods for the model reduction and control of a pilot plant evaporator.

Figure 6.1 shows how a reduced order discrete-time model and a reduced order control law can be derived from a high order continuous-time model. The desired reduced order model and control law are to be in the discrete-time form for ease in computer simulation and control system application using an on-line computer.

The model reduction paths are numbered 1 and 4 in Figure 6.1. Most of the literature deals with the continuous-time model reduction of path 1 with work being done in the following areas:

a) Reduction of a state space model using a modal analysis of the system. This topic is discussed later in this chapter.

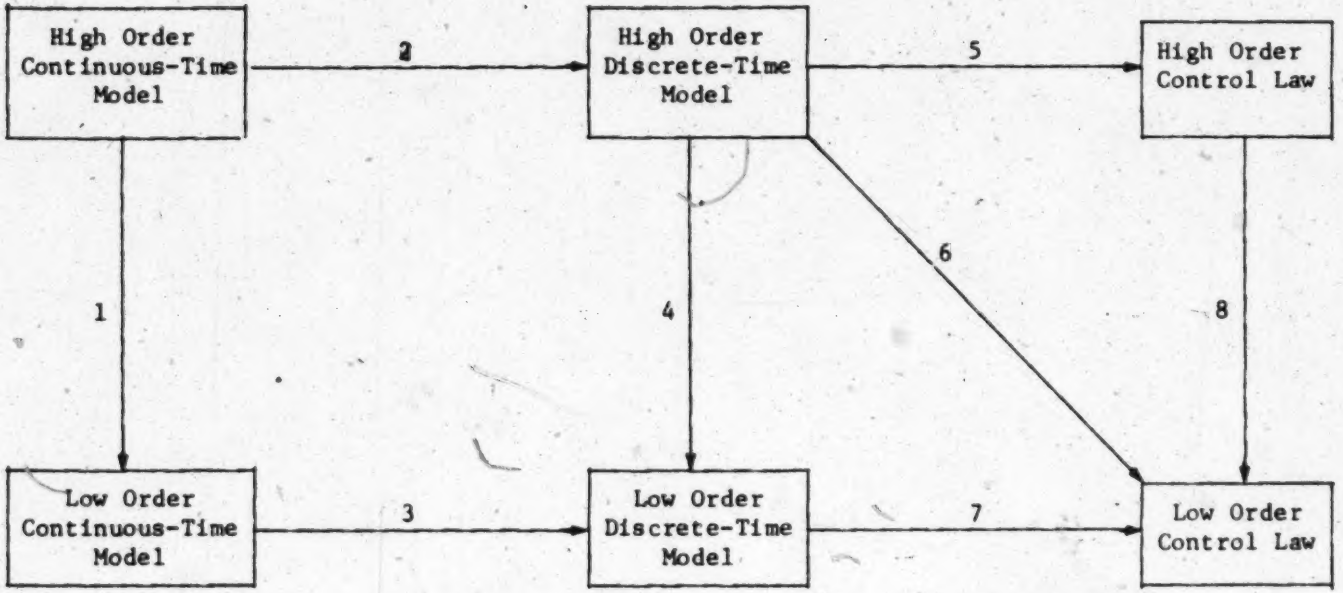


FIGURE 6.1 PATHS FOR THE CALCULATION OF LOW ORDER MODELS AND CONTROL LAWS FROM HIGH ORDER MODELS

b) Reduction of a state space model by minimizing some measure of difference between the high order and the low order models. One of these methods involves a least squares analysis which was discussed in detail in Chapter Four.

c) Reduction of a transfer function model. This has been discussed using different approaches by Chen and Shieh [4], Gibilaro and Lees [20], Sinha and Bereznai [35], Hsia [23] and Fellows et al [18].

The reduction of discrete-time models (path 4) was discussed in Chapter Three using a modal analysis and in Chapter Four using a least squares analysis. One form of the least squares approach is applied to the evaporator model later in this chapter.

The various methods of calculating a reduced order control law from a high order discrete-time model shown in Figure 6.1 are:

a) Model reduction followed by control system design based only on the reduced order model (path 4 followed by path 7). This has been considered by Nicholson [31 - 33] and Anderson [1] for state space models and Bereznai and Sinha [2] and Chen and Shieh [3] for transfer function models and is also considered in this chapter.

b) Model reduction with the added restriction that the optimal control law for this reduced order model must be the "best" suboptimal control law for the original high order system. This involves paths 4 and 7 at the same time. Rogers and Sworder [34] utilize this approach.

c) Generation of the reduced order control system directly from the high order, open-loop model, without calculating a reduced order model. (Path 6) This approach has been considered by Dabke [7,8], Levine and Athans [26], Kosut [25] and Davison et al. [9,10,15-17].

d) Calculation of the reduced order control law from a high order control system designed for the original high order model. (Path 5 followed by path 8). Kosut [25] requires the optimal control law for two simple forms of his controller. One result using this approach was derived in Chapter Five and is compared to other approaches later in this chapter.

The rest of this chapter is organized as follows. Section 6.2 discusses the modal approach to model reduction in more detail. It also summarizes the model reduction and reduced order control law design techniques which are applied to the pilot plant evaporator. Section 6.3 discusses the model reduction results and Section 6.4 discusses the application of the reduced order control laws to the control of the pilot plant evaporator.

6.2 MODEL REDUCTION AND CONTROL LAW DESIGN METHODS TO BE APPLIED TO THE EVAPORATOR

This section discusses the methods of model reduction and reduced order control law design which are applied to the evaporator (described in Appendix A) later in this chapter. The general form of the model and control law will first be discussed. These are shown

in Figure 6.1a, in which the model and control laws have been added to the basic structure of Figure 6.1. The nomenclature in Figure 6.1a include:

$\underline{x}$ = n-dimensional state vector

$\underline{x}_1$ = ℓ-dimensional subset of $\underline{x}$ which will be retained in the low order model or control law

$\underline{x}_2$ = $(n-\ell)$-dimensional subset of $\underline{x}$

$\underline{x}_R$ = ℓ-dimensional state vector of the reduced order model

$\hat{\underline{u}}$ = $(m+q)$-dimensional input vector which can be partitioned into an m-dimensional control vector, $\underline{u}$, and a q-dimensional disturbance vector $\underline{d}$, as

$$\hat{\underline{u}} = \begin{bmatrix} \underline{u} \\ \underline{d} \end{bmatrix} \tag{6.11}$$

$\underline{y}$ = p-dimensional output vector

$\underline{y}^{SP}$ = a constant setpoint vector such that y_i^{SP} is the desired value of y_i, i = 1, ..., p

$\underline{A}$, $\hat{\underline{B}}$, $\underline{C}$, $\hat{\underline{\Delta}}$ = constant coefficient matrices of appropriate dimensions representing the process. The input forcing matrices $\hat{\underline{B}}$ and $\hat{\underline{\Delta}}$ can be partitioned to agree with the partitioning of $\hat{\underline{u}}$ as

$$\hat{\underline{B}} = (\underline{B}, \underline{D}) \tag{6.12}$$

$$\hat{\underline{\Delta}} = (\underline{\Delta}, \underline{\Theta}) \tag{6.13}$$

j = counter for time intervals such that $\underline{x}(j)$ denotes $\underline{x}(t)$ at $t = jT$, where T is the discrete time interval.

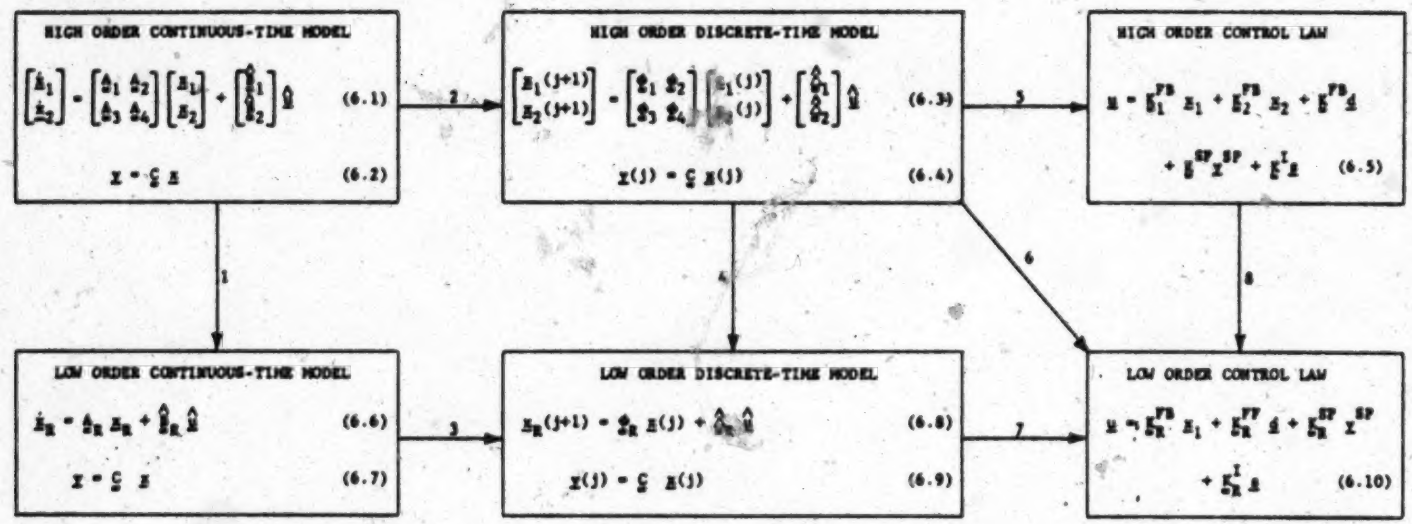


FIGURE 6.1a BASIC FORMS OF THE MODELS AND CONTROL LAWS

$\underline{K}_1^{FB}$, $\underline{K}_2^{FB}$, $\underline{K}^{FF}$, $\underline{K}^{SP}$, $\underline{K}^{I}$ = constant coefficient matrices of appropriate dimensions representing the parameters of the control law

$\underline{s}$ = integral contribution to the control law represented for continuous-time systems as

$$\underline{s} = \int_0^t (\underline{y} - \underline{y}^{SP})\, dt \tag{6.14}$$

or for discrete-time systems as

$$\underline{s}(j) = \sum_{i=0}^{j} (\underline{y}(i) - \underline{y}^{SP}) \tag{6.15}$$

The subscript, R, refers to a reduced order vector or matrix and the matrices $\hat{\underline{\underline{B}}}_R$ and $\hat{\underline{\underline{\Delta}}}_R$ can be partitioned (as were $\hat{\underline{\underline{B}}}$ and $\hat{\underline{\underline{\Delta}}}$) into

$$\hat{\underline{\underline{B}}}_R = (\underline{\underline{B}}_R, \underline{\underline{D}}_R) \tag{6.16}$$

$$\hat{\underline{\underline{\Delta}}}_R = (\underline{\underline{\Delta}}_R, \underline{\underline{\Theta}}_R) \tag{6.17}$$

6.2.1 Modal Approach to Model Reduction

The modal methods of model reduction are those which retain specific modes (or eigenvalues) of the original high order model in the reduced order model. These methods make use of the modal analysis of the high order system as follows. A matrix $\underline{\underline{M}}$ exists such that

$$\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} = \begin{bmatrix} \underline{\underline{M}}_1 & \underline{\underline{M}}_2 \\ \underline{\underline{M}}_3 & \underline{\underline{M}}_4 \end{bmatrix} \begin{bmatrix} \underline{z}_1 \\ \underline{z}_2 \end{bmatrix} \tag{6.18}$$

which will transform Equation (6.1) into its Jordan canonical form

$$\begin{bmatrix} \dot{\underline{z}}_1 \\ \dot{\underline{z}}_2 \end{bmatrix} = \begin{bmatrix} \underline{J}_1 & \underline{0} \\ \underline{0} & \underline{J}_2 \end{bmatrix} \begin{bmatrix} \underline{z}_1 \\ \underline{z}_2 \end{bmatrix} + \begin{bmatrix} \hat{\underline{G}}_1 \\ \hat{\underline{G}}_2 \end{bmatrix} \hat{\underline{u}} \tag{6.19}$$

and Equation (6.3) into its Jordan canonical form

$$\begin{bmatrix} \underline{z}_1(j+1) \\ \underline{z}_2(j+1) \end{bmatrix} = \begin{bmatrix} \underline{\alpha}_1 & \underline{0} \\ \underline{0} & \underline{\alpha}_2 \end{bmatrix} \begin{bmatrix} \underline{z}_1(j) \\ \underline{z}_2(j) \end{bmatrix} + \begin{bmatrix} \hat{\underline{\delta}}_1 \\ \hat{\underline{\delta}}_2 \end{bmatrix} \hat{\underline{u}} \tag{6.20}$$

where $\underline{z}$ = n-dimensional canonical state vector partitioned into an ℓ-dimensional vector $\underline{z}_1$ and an $(n - \ell)$-dimensional vector $\underline{z}_2$

$$\underline{J} = \begin{bmatrix} \underline{J}_1 & \underline{0} \\ \underline{0} & \underline{J}_2 \end{bmatrix} = \underline{V}\,\underline{A}\,\underline{M} \tag{6.21}$$

$$\underline{\alpha} = \begin{bmatrix} \underline{\alpha}_1 & \underline{0} \\ \underline{0} & \underline{\alpha}_2 \end{bmatrix} = \underline{V}\,\underline{\Phi}\,\underline{M} \tag{6.22}$$

$$\hat{\underline{G}} = \begin{bmatrix} \hat{\underline{G}}_1 \\ \hat{\underline{G}}_2 \end{bmatrix} = \underline{V}\,\hat{\underline{B}} \tag{6.23}$$

$$\hat{\underline{\delta}} = \begin{bmatrix} \hat{\underline{\delta}}_1 \\ \hat{\underline{\delta}}_2 \end{bmatrix} = \underline{V}\,\hat{\underline{\Delta}} \tag{6.24}$$

$$\underline{V} = \begin{bmatrix} \underline{V}_1 & \underline{V}_2 \\ \underline{V}_3 & \underline{V}_4 \end{bmatrix} = \underline{M}^{-1} \tag{6.25}$$

Matrices $\hat{\underline{G}}$ and $\hat{\underline{\delta}}$ can be partitioned (as were $\hat{\underline{B}}$ and $\hat{\underline{\Delta}}$) into

$$\hat{\underline{\underline{G}}} = (\underline{\underline{G}}, \underline{\underline{H}}) \tag{6.26}$$

$$\hat{\underline{\underline{\delta}}} = (\underline{\underline{\delta}}, \underline{\underline{\gamma}}) \tag{6.27}$$

Even though there have been many papers written on the modal approach to model reduction, most of the methods produce one of two basic results or two modified forms of one of these basic results. Most of the modal methods calculate the same $\underline{\underline{A}}_R$ or $\underline{\underline{\Phi}}_R$. They differ only in the approach taken and in the resulting $\hat{\underline{\underline{B}}}_R$ and $\hat{\underline{\underline{\Delta}}}_R$. Most of the comments to follow will refer specifically to the continuous-time reduction methods used to obtain Equation (6.6) from Equation (6.1).

One of the basic modal approaches was presented in different forms by Marshall [27] and Chidambara [5] (Chidambara's method marked C2). The equivalence of the two formulations was noted by Graham [21]. They assume that the modes which do not contribute significantly to the responses of the states of interest, reach their final steady state values immediately after a disturbance enters the system. The main advantage of this result is that it provides a reduced order model with exact steady state agreement with the original high order model for step disturbances. Davison [5,11], however, objects to this model because the retained modes are not excited in the same proportions in the reduced order model as they are in the high order model. The equivalence of these results is shown in Appendix C.

The second basic modal approach was presented in slightly different forms by Nicholson [33] and Davison [13]. The equivalence

of their formulations is shown in Appendix C. Their resulting reduced order model has its modes excited in the same proportions as they were in the high order model. The main disadvantage of this result is that a steady state error exists between the high order and the low order responses for sustained disturbances. Davison [11] claims that this error will be small if the order of reduction is not large.

Several different formulations have been presented in the literature for modifications to the result of Davison and Nicholson to provide steady state agreement. These, however, all reduce to only two different results.

Davison [12] presented one of these revisions; each state variable of the reduced order model is adjusted by a ratio of the desired steady state to the steady state obtained for the unmodified reduced order model. For systems with a single input, Davison's modified result can be arranged into the standard form of Equation (6.6). His modification alters both $\underline{\underline{A}}_R$ and $\hat{\underline{\underline{B}}}_R$ but retains the eigenvalues. However, if there is more than one input, this arrangement is not possible. Also, his modification cannot be applied to models with a singular $\underline{\underline{A}}$ or $\underline{\underline{A}}_R$ (unmodified form using Davison's basic method mentioned above) since $\underline{\underline{A}}^{-1}$ and $\underline{\underline{A}}_R^{-1}$ are required. However, in Davison's example, both $\underline{\underline{A}}$ and $\underline{\underline{A}}_R$ are singular. He gets around this problem by changing the parameters in his original model slightly, so that $\underline{\underline{A}}$ is no longer singular [14].

The second revision to Davison's basic method has been presented by Davison [5, 11] and by Fossard [19]. This result was also

obtained by Chidambara [6] (Chidambara's method marked C1) and Graham [21], not as revisions to Davison's basic method, but in independent derivations. The equivalence of these results is derived and discussed in Appendix C. This revision can be written in two equivalent forms as:

$$\dot{\underline{x}}_\ell = \underline{\underline{A}}_R \underline{x}_\ell + \hat{\underline{\underline{B}}}^D_R \hat{\underline{u}} \tag{6.28}$$

$$\underline{x}_R = \underline{x}_\ell + \underline{\underline{E}}_R \hat{\underline{u}} \tag{6.29}$$

or as

$$\dot{\underline{x}}_R = \underline{\underline{A}}_R \underline{x}_R + \hat{\underline{\underline{B}}}^M_R \hat{\underline{u}} + \underline{\underline{E}}_R \dot{\hat{\underline{u}}} \tag{6.30}$$

where $\underline{x}_\ell$ = a general ℓ-dimensional vector

$\hat{\underline{\underline{B}}}^D_R = \hat{\underline{\underline{B}}}_R$ using Davison's basic method

$\hat{\underline{\underline{B}}}^M_R = \hat{\underline{\underline{B}}}_R$ using Marshall's basic method

It can be seen from Equation (6.30) that this result reduces to Marshall's result for the case of $\dot{\hat{\underline{u}}} = \underline{0}$.

Even though all four of these results are equivalent, the form presented by Davison [5,11] cannot be used when either $\underline{\underline{A}}$ or $\underline{\underline{A}}_R$ is singular.

Graham [21] discusses the idea of applying this method with a non-zero initial vector for $\underline{x}_R$ in Equation (6.30). His initial value vector is

$$\underline{x}_R(0^+) = \underline{\underline{M}}_1 \left\{ \underline{\underline{V}}_1 \underline{x}_1(0^+) + \underline{\underline{V}}_2 \underline{x}_2(0^+) \right\} + \underline{\underline{E}}_R \hat{\underline{u}}(0^+) \tag{6.31}$$

This modification provides better agreement with the high order model earlier in the response but with the disadvantage that there is

a large error at the start of the trajectory. If the model were to be used for control purposes, the initial time error could not be accepted. In the formulation of Equations (6.28) and (6.29) the vector $\underline{x}_\ell$, at time zero, can be set arbitrarily. Thus, in the examples in Section 6.3.1, to ensure that $\underline{x}_R(0) = \underline{x}_1(0)$, the initial value of $\underline{x}_\ell$ will be specified, using Equation (6.29), to be

$$\underline{x}_\ell(0) = \underline{x}_1(0) - \underline{\underline{E}}_R \hat{\underline{u}}(0) \tag{6.32}$$

The modal methods which are applied to the evaporator model in this chapter are those presented by Marshall [27], Davison [13] and Fossard [19] for the reduction of continuous-time models and extended for use with discrete-time models in Chapter Three. Each of these methods produce the same $\underline{\underline{A}}_R$ and $\underline{\underline{\Phi}}_R$ as

$$\underline{\underline{A}}_R = \underline{\underline{M}}_1 \underline{\underline{A}}_1 \underline{\underline{M}}_1^{-1} = \underline{\underline{A}}_1 - \underline{\underline{A}}_2 \underline{\underline{V}}_4^{-1} \underline{\underline{V}}_3 = \underline{\underline{A}}_1 + \underline{\underline{A}}_2 \underline{\underline{M}}_3 \underline{\underline{M}}_1^{-1} \tag{6.33}$$

and

$$\underline{\underline{\Phi}}_R = \underline{\underline{M}}_1 \underline{\underline{\alpha}}_1 \underline{\underline{M}}_1^{-1} = \underline{\underline{\Phi}}_1 - \underline{\underline{\Phi}}_2 \underline{\underline{V}}_4^{-1} \underline{\underline{V}}_3 = \underline{\underline{\Phi}}_1 + \underline{\underline{\Phi}}_2 \underline{\underline{M}}_3 \underline{\underline{M}}_1^{-1} \tag{6.34}$$

These equivalences can be shown using relations in Appendix D. Expressions for $\hat{\underline{\underline{B}}}_R$, $\underline{\underline{E}}_R$ and $\hat{\underline{\underline{\Delta}}}_R$ for each of these methods are presented in Table 6.1.

In the application of these methods, $\underline{x}_\ell(0)$ is calculated using Equation (6.32). Also, no reference will be made to indicate whether the continuous-time model or the discrete-time model was

TABLE 6.1

MATRICES $\hat{\underline{B}}_R$, $\hat{\underline{\Delta}}_R$ AND $\underline{E}_R$ FOR REDUCED ORDER MODELS

OBTAINED USING A MODAL ANALYSIS

Approach Taken By (see Note 3)	Continuous-Time Model		Discrete-Time Model (See Note 1)	
	$\hat{\underline{B}}_R$	$\underline{E}_R$	$\hat{\underline{\Delta}}_R$	$\underline{E}_R$
Marshall [27]	$\hat{\underline{B}}_1 - \underline{A}_2 \underline{V}_4^{-1} \underline{J}_2^{-1} \hat{\underline{G}}_2$	$\underline{0}$	$\hat{\underline{\Delta}}_1 + \underline{\Phi}_2 \underline{V}_4^{-1} (\underline{I} - \underline{\alpha}_2)^{-1} \hat{\underline{\delta}}_2$	$\underline{0}$
Davison [13]	$\underline{M}_1 \hat{\underline{G}}_1$	$\underline{0}$	$\underline{M}_1 \hat{\underline{\delta}}_1$	$\underline{0}$
Fossard [19]	$\underline{M}_1 \hat{\underline{G}}_1$	$-\underline{M}_2 \underline{J}_2^{-1} \hat{\underline{G}}_2$ (See Note 2)	$\underline{M}_1 \hat{\underline{\delta}}_1$	$\underline{M}_2 (\underline{I} - \underline{\alpha}_2)^{-1} \hat{\underline{\delta}}_2$ (See Note 2)

Notes: 1. The discrete-time results have been derived in Chapter Three.

2. It is shown in Chapter Three that these two expressions for $\underline{E}_R$ are identical.

3. Matrices $\underline{A}_R$ and $\underline{\Phi}_R$ for all methods are identical and are shown by Equations (6.33) and (6.34).

reduced, since, as shown in Chapter Three, the same discrete-time model is obtained.

6.2.2 Least Squares Approach to Model Reduction

The application of least squares to model reduction falls within the general category of methods which minimize a function of the difference between the response of the high order model and that of the low order model. Specifically, least squares minimizes the sum of the squares of the difference between the high order model response and resulting reduced order model response for all the states in the reduced order model.

In this approach the transition matrix, $\underline{\underline{\Phi}}_R$, is calculated using least squares while $\hat{\underline{\underline{\Delta}}}_R$ is calculated to give agreement between the high order and reduced order models after the initial transient dies out. The high order model and a sequence of uniformly distributed random numbers are used to calculate the data for use with least squares, as discussed in Chapter Four. The results are summarized here for the reduction of discrete-time models with at least one integrating state (that is, an eigenvalue of $\underline{\underline{\Phi}}$ equal to one).

If the reduced order state vector $\underline{x}_R$ is partitioned as

$$\underline{x}_R = \begin{bmatrix} \underline{x}_I \\ \underline{x}_N \end{bmatrix} \tag{6.35}$$

corresponding to the integrating, $\underline{x}_I$, and the non-integrating, $\underline{x}_N$, state variables of $\underline{x}_R$, then $\underline{\underline{\Phi}}_R$ can similarly be partitioned as,

$$\underline{\underline{\Phi}}_R = (\underline{\underline{\Phi}}_I, \underline{\underline{\Phi}}_N) \tag{6.36}$$

The partition of $\underline{\Phi}_R$ corresponding to the integrating state variables can be pre-specified as

$$\underline{\Phi}_I = \begin{bmatrix} \underline{I} \\ \underline{0} \end{bmatrix} \tag{6.37}$$

The dimension of the identity matrix in Equation (6.37) is the same as the number of integrating state variables. The partition of $\underline{\Phi}_R$ corresponding to the non-integrating states is calculated using least squares as

$$\underline{\Phi}_N^T = (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T \underline{W} \tag{6.38}$$

where

$$\underline{Z} = \begin{bmatrix} \underline{x}_N^T (1) \\ \underline{x}_N^T (2) \\ \vdots \\ \underline{x}_N^T (k) \end{bmatrix} \quad \text{and} \quad \underline{W} = \begin{bmatrix} \underline{w}^T (1) \\ \underline{w}^T (2) \\ \vdots \\ \underline{w}^T (k) \end{bmatrix} \tag{6.39}$$

and where k is the number of data vectors. The j^{th} row of $\underline{W}$ is defined as

$$\underline{w}(j) = \underline{\Phi}_{1N} \underline{x}_N(j) + \underline{\Phi}_2 \underline{x}_2(j) \tag{6.40}$$

where $\underline{\Phi}_1$ has been partitioned as

$$\underline{\Phi}_1 = (\underline{\Phi}_{1I}, \underline{\Phi}_{1N}) \tag{6.41}$$

corresponding to the partitioning of $\underline{x}_R$ in Equation (6.35). The elements of vectors $\underline{x}_N$ and $\underline{x}_2$ are specified as random numbers.

After $\underline{\Phi}_R$ has been obtained, $\hat{\underline{\Delta}}_R$ is calculated to ensure that the reduced order and high order models give the same response for large values of the counter, j. Thus,

$$\hat{\underline{\Delta}}_R^i = \frac{\underline{x}_R^i\,(j+1) - \underline{\Phi}_R\,\underline{x}_R^i\,(j)}{\hat{u}_i} \qquad (6.42)$$

where $\hat{\underline{\Delta}}_R^i$ = i^{th} column of $\hat{\underline{\Delta}}_R$

$\hat{u}_i$ = i^{th} element of $\hat{\underline{u}}$

$\underline{x}_R^i$ = $\underline{x}_R$ for large values of the counter j for a step change in $\hat{u}_i$. ($\hat{u}_k = 0,\ k \neq i$)

The values of the elements of $\underline{x}_R^i\,(j+1)$ and $\underline{x}_R^i\,(j)$ corresponding to the non-integrating states are equivalent and are the steady state values of these states for the non-zero value of $\hat{u}_i$, ($\hat{u}_k = 0,\ k \neq i$), as calculated from the high order model. The values of the elements of $\underline{x}_R^i\,(j+1)$ corresponding to the integrating states are calculated as

$$\underline{x}_R^i(j+1) = \underline{\Phi}_1\,\underline{x}_R^i(j) + \underline{\Phi}_2\,\underline{x}_2(j) + \hat{\underline{\Delta}}_1^i\,\hat{u}_i \qquad (6.43)$$

where $\hat{\underline{\Delta}}_1^i$ = i^{th} column of $\hat{\underline{\Delta}}_1$, and the elements of $\underline{x}_R^i(j)$ corresponding to the integrating states are specified by random numbers and the elements of $\underline{x}_R^i(j)$ corresponding to the non-integrating states are the above mentioned steady state values for the same non-zero value of $\hat{u}_i$.

Equation (6.42) provides for steady state agreement of the non-integrating states and agreement of the rate of change of the integrating states after the transients die out.

6.2.3 Approaches Used for Reduced Order Control Law Design

The modal analysis and the least squares analysis (using

random data) were used to design reduced order control laws using the following paths shown in Figure 6.1:

a) Path 4 followed by path 7 - the calculation of a reduced order model and the design of the reduced order control law directly from this model using standard techniques.

b) Path 5 followed by path 8 - the calculation of the high order optimal control law and the reduction of this controller to form the reduced order control law.

In the first approach, the reduced order control law is calculated from a reduced order model. The modal approach to model reduction used in this step was that of Marshall as described in Chapter Three and by Marshall [27] and was chosen from the modal methods discussed in Section 6.2.1, since it was the only method whose result could be arranged in the standard form of Equation (6.8) and which guaranteed steady state agreement between the high order and the reduced order models. The least squares approach used in this step is that outlined in Section 6.2.2. It calculates $\underline{\underline{\Phi}}_R$ using random data and $\hat{\underline{\underline{\Delta}}}_R$ for agreement at large times. Once the required reduced order model is obtained, the desired reduced order control law is calculated using any applicable method. In this work the optimal control approach outlined by Newell [28] was used.

In the second approach, the reduced order control law is calculated from the high order control law for the system. A modal analysis and a least squares analysis (with random data) are used to eliminate selected states from the high order control law. The modal approach is that which is discussed in detail in Chapter Five

and so it will not be discussed further here. The least squares approach is similar to that applied to model reduction in Chapter Four. The reduced order control law is calculated by fitting the response of the reduced order controller to the response of the high order controller. Appendix E shows that the same reduced order controllers can be obtained from two different (but equivalent) formulations, as:

a) Calculating the control matrices so that the control vector, $\underline{u}$, as calculated with the reduced order control law, agrees with that calculated by the high order control law.

b) Calculating the control matrices so that the closed-loop response of the high order system, when controlled by the reduced order control law, agrees with that obtained when controlled by the high order control law.

The resulting reduced order control matrices are calculated from

$$\underline{\underline{K}}_R^T = (\underline{\underline{Z}}^T \underline{\underline{Z}})^{-1} \underline{\underline{Z}}^T \underline{\underline{W}} \tag{6.44}$$

where $\underline{\underline{K}}_R$ = the control matrices to be calculated. In the case of proportional feedback, feedforward, setpoint, integral feedback control,

$$\underline{\underline{K}}_R = (\underline{\underline{K}}_R^{FB}, \underline{\underline{K}}_R^{FF}, \underline{\underline{K}}_R^{SP}, \underline{\underline{K}}_R^{I}) \tag{6.45}$$

$$\underline{\underline{Z}} = \begin{bmatrix} \underline{z}^T(1) \\ \underline{z}^T(2) \\ \vdots \\ \underline{z}^T(k) \end{bmatrix} \quad \text{and} \quad \underline{\underline{W}} = \begin{bmatrix} \underline{w}^T(1) \\ \underline{w}^T(2) \\ \vdots \\ \underline{w}^T(k) \end{bmatrix} \tag{6.46}$$

where k is the number of data vectors. The vectors $\underline{z}(j)$ which make up the rows of $\underline{\underline{Z}}$, for the case of proportional feedback, feedforward, setpoint, integral feedback control, represent the vectors $\underline{x}_1(j)$, $\underline{d}(j)$, $\underline{y}^{SP}$ and $\underline{s}(j)$ so that

$$\underline{z}(j) = \begin{bmatrix} \underline{x}_1\ (j) \\ \underline{d}\ (j) \\ \underline{y}^{SP} \\ \underline{s}\ (j) \end{bmatrix} \tag{6.47}$$

The vectors $\underline{w}(j)$ which make up the rows of $\underline{\underline{W}}$, for the case of proportional feedback, feedforward, setpoint, and integral feedback control, are calculated as

$$\underline{w}(j) = \underline{\underline{K}}_1^{FB}\ \underline{x}_1(j) + \underline{\underline{K}}_2^{FB}\ \underline{x}_2(j) + \underline{\underline{K}}^{FF}\ \underline{d}(j) + \underline{\underline{K}}^{SP}\ \underline{y}^{SP} + \underline{\underline{K}}^{I}\ \underline{s}(j) \tag{6.48}$$

The values of $\underline{x}_1(j)$, $\underline{x}_2(j)$, $\underline{d}(j)$, $\underline{y}^{SP}$ and $\underline{s}(j)$ in Equations (6.47) and (6.48) are specified as a sequence of uniformly distributed random numbers.

In the application of the result, if a control law which does not include all four control modes discussed in the above analysis is desired, then the terms in Equations (6.45), (6.47) and (6.48) corresponding to the undesired control matrices are simply left out of the analysis. For the example of proportional feedback control only, Equations (6.45), (6.47) and (6.48) will appear as:

$$\underline{\underline{K}}_R = \underline{\underline{K}}_R^{FB}$$

$$\underline{z}(j) = \underline{x}_1(j) \tag{6.49}$$

$$\underline{w}(j) = \underline{\underline{K}}_1^{FB}\, \underline{x}_1(j) + \underline{\underline{K}}_2^{FB}\, \underline{x}_2(j)$$

6.3 REDUCTION OF AN EVAPORATOR MODEL

The model reduction methods discussed in Sections 6.2.1 and 6.2.2 have been applied to the reduction of the tenth order evaporator model described in Appendix A. These results are discussed in this section. The various reduced order models calculated are shown in Appendix F. Most of the reduced order models discussed in this section are third order models in which the reduced order state vector $\underline{x}_R = [W1, W2, C2]^T$. These are the three process variables which are of most interest since, as is discussed in Section 6.4, they are the process variables which must be controlled.

6.3.1 Modal Methods

The modal methods applied to the reduction of the evaporator model are those presented in Table 6.1.

Figure 6.2 compares the time domain response to a +20% step change in feedflow using various third order open loop models and the tenth order open loop model. It shows that the third order model calculated using Davison's method has considerable steady state error. (It should be pointed out, however, that the fifth order model calculated using Davison's method has negligible steady state error.) Figure 6.2 also shows that the third order models calculated using Marshall's method and Fossard's method are identical, lead the

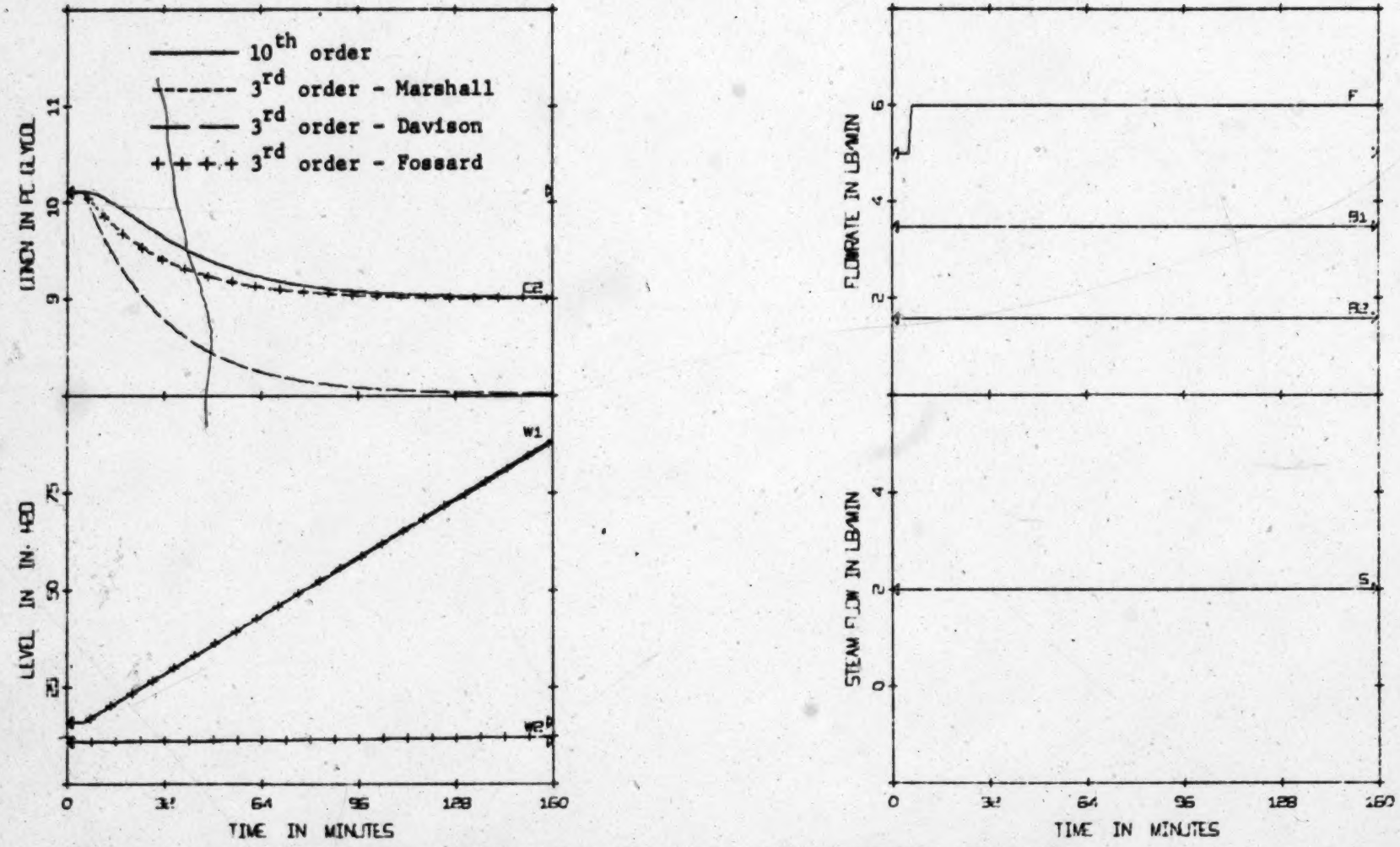


FIGURE 6.2 COMPARISON OF THIRD ORDER MODELS OBTAINED BY MODAL REDUCTION APPROACHES (10L(2018), 3L(2021), 3L(2065A), 3L(2094)/D, +20%F/OL/2018, 2021, 2065A, 2097)

tenth order model slightly, and result in zero steady state error. The models calculated by Marshall's method and Fossard's method yield the same time domain response since the input is constant (+20% step in feedflow, open loop). In order to have zero initial error using Fossard's method, the initial values of the elements of $\underline{x}$ were set, using Equation (6.32) to

$$\underline{x}_{\ell}(0) = \underline{x}_1(0) - \underline{E}_R \hat{\underline{u}}(0) = \begin{bmatrix} W1(0) \\ W2(0) \\ C2(0) \end{bmatrix} = \begin{bmatrix} 0.0014 \\ 0.0024 \\ -0.097 \end{bmatrix} \qquad (6.50)$$

Figure 6.3 compares the response of the third order model calculated using Fossard's method when $\underline{x}_R(0) = \underline{x}_1(0)$ and when $\underline{x}_R(0)$ is defined by Equation (6.31). The response with $\underline{x}_R(0) \neq \underline{x}_1(0)$ converges to the high order model response faster than the other response. However, it has a large initial error (at zero time) in C2, which is undesirable. Also, for this example, the response with $\underline{x}_R(0) = \underline{x}_1(0)$ is satisfactory since the error is not large anywhere along the response.

Figure 6.4 demonstrates that the agreement with the tenth order model becomes better as the order of the reduced order model is increased, as would be expected. The response of the fifth order model, with $\underline{x}_R^T = [W1, C1, H1, W2, C2]$, agrees better with the response of the tenth order model than does that of the third order model.

Figure 6.5 demonstrates that the eigenvalues to be retained in the reduced order model must be carefully selected by showing the response of two fourth order models with $\underline{x}_R^T = [W1, H1, W2, C2]$. When

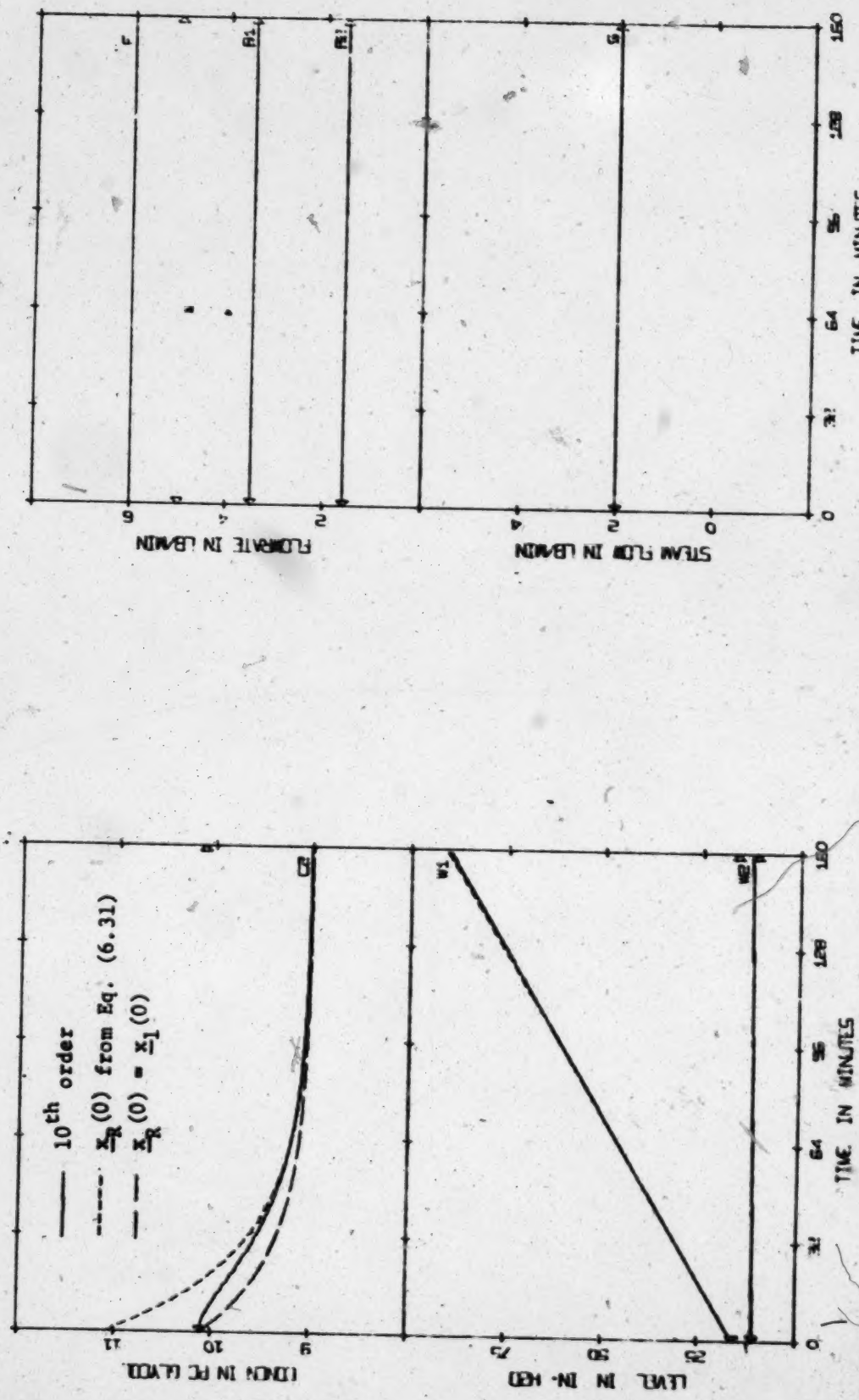


FIGURE 6.3 COMPARISON OF FOSSARD'S THIRD ORDER MODEL FOR DIFFERENT $x_R(0)$ (10L(2018), 3L(2094), 3L(2094)/D, +20%F/OL/2018, 2097/2094)

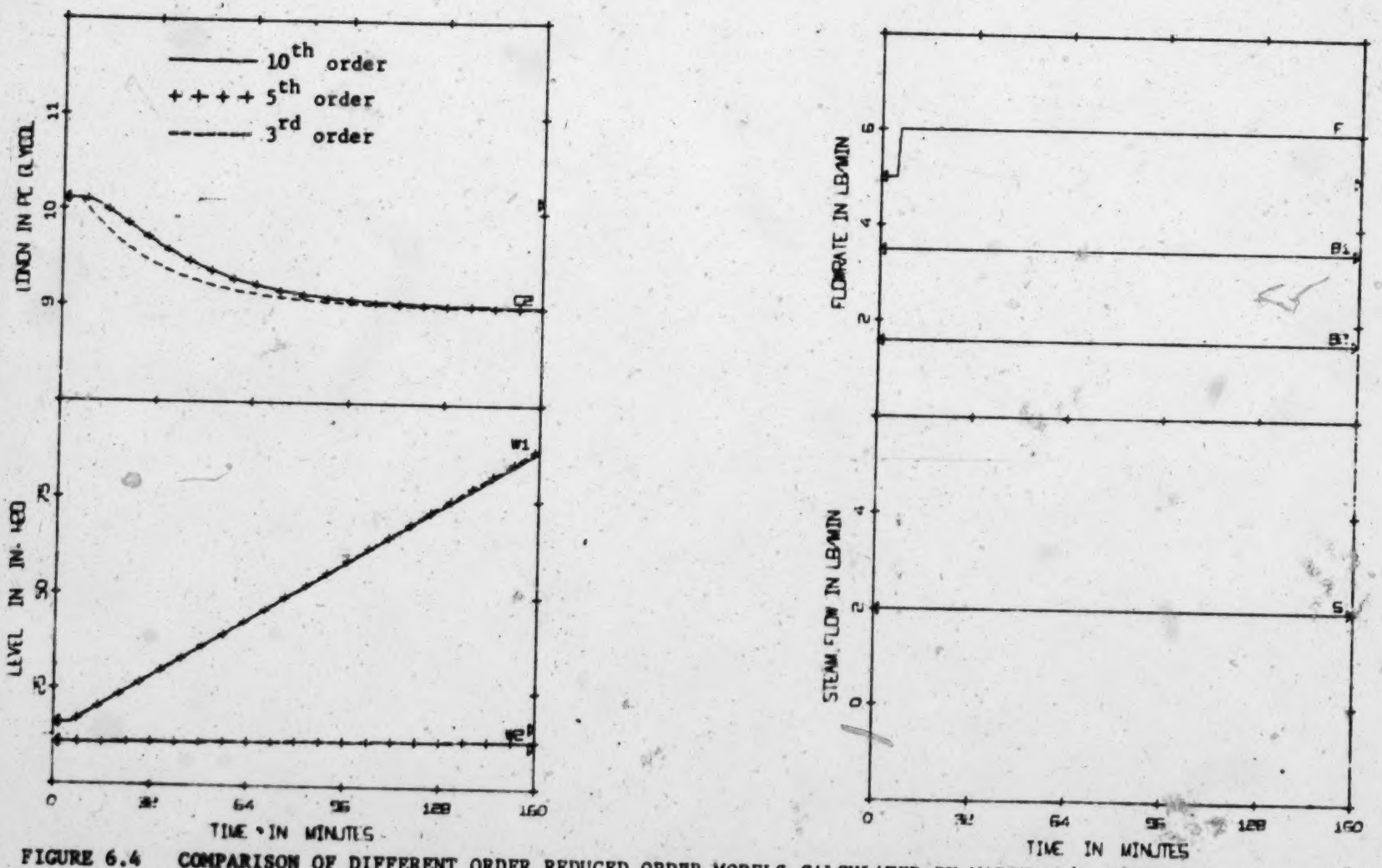
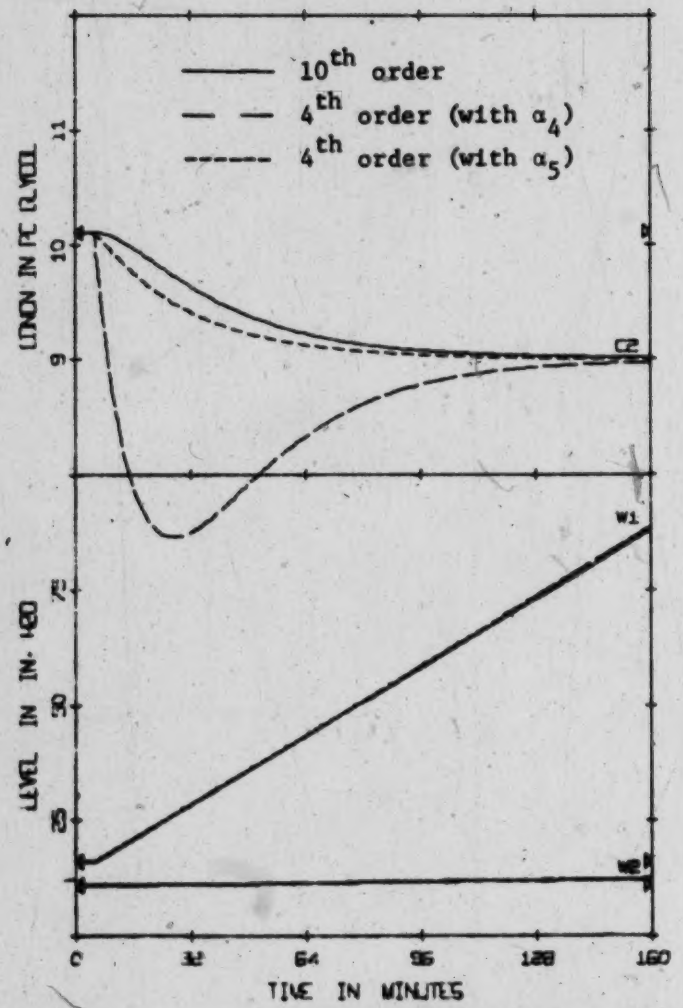


FIGURE 6.4 COMPARISON OF DIFFERENT ORDER REDUCED ORDER MODELS CALCULATED BY MARSHALL'S APPROACH (10L(2018), 5L(2020), 3L(2021)/D, +20%F/OL/2018, 2020, 2021)



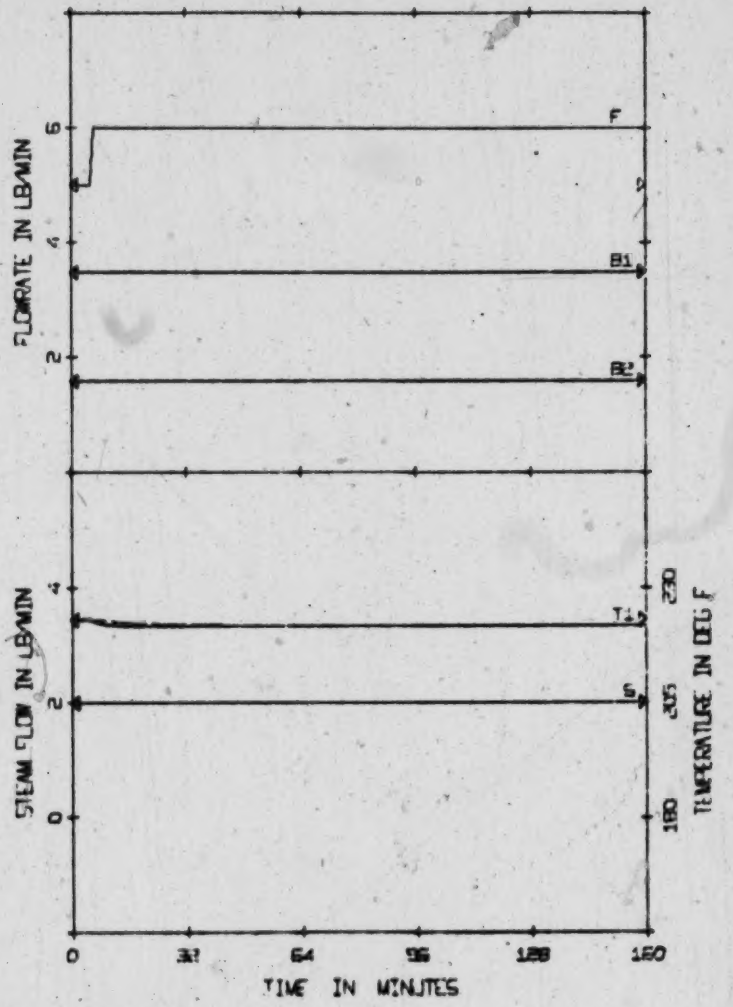


FIGURE 6.5 COMPARISON OF FOURTH ORDER MODELS RETAINING DIFFERENT EIGENVALUES (10L(2018), 4L(2059A), 4L(2060A)/D, +20%F/0L/2018, 2059A, 2060A)

the four eigenvalues retained in $\underline{\Phi}_R$ were those of $\underline{\Phi}$ which were closest to one, poor agreement in the C2 response resulted. The modal matrix for the tenth order model in Appendix A, shows that none of these four largest eigenvalues affect H1 (x(5)) significantly. However, the fifth largest eigenvalue does affect H1 significantly. Thus, when the fifth largest eigenvalue (0.7354) replaces the fourth largest eigenvalue (0.9212), much better results were obtained.

6.3.2 Least Squares Method

Figure 6.6 shows the comparison of the least squares approach with the modal approach of Marshall. The response of the third order model obtained by the least squares approach agrees slightly better with that of the high order model than does the response of the third order model obtained by the modal analysis. Table 6.2 shows that the third eigenvalue of $\underline{\Phi}_R$ from the least squares model is larger than the corresponding eigenvalue of $\underline{\Phi}_R$ from the modal model. This larger eigenvalue causes the slower response and the better agreement.

6.4 REDUCED ORDER CONTROL OF A PILOT PLANT EVAPORATOR

The reduced order control laws discussed in Section 6.2.3 were applied to the pilot plant evaporator described in Appendix A. The objective was the lowest order control law which would still satisfactorily stabilize the system, The result was a third order control law with vector $\underline{x}_1$, containing W1, W2 and C2. The product concentration, C2, must be included in $\underline{x}_1$ since regulation of C2 is the primary control objective. W1 and W2 must be included since these are integrating states (corresponding to eigenvalues of $\underline{\Phi}$ equal to

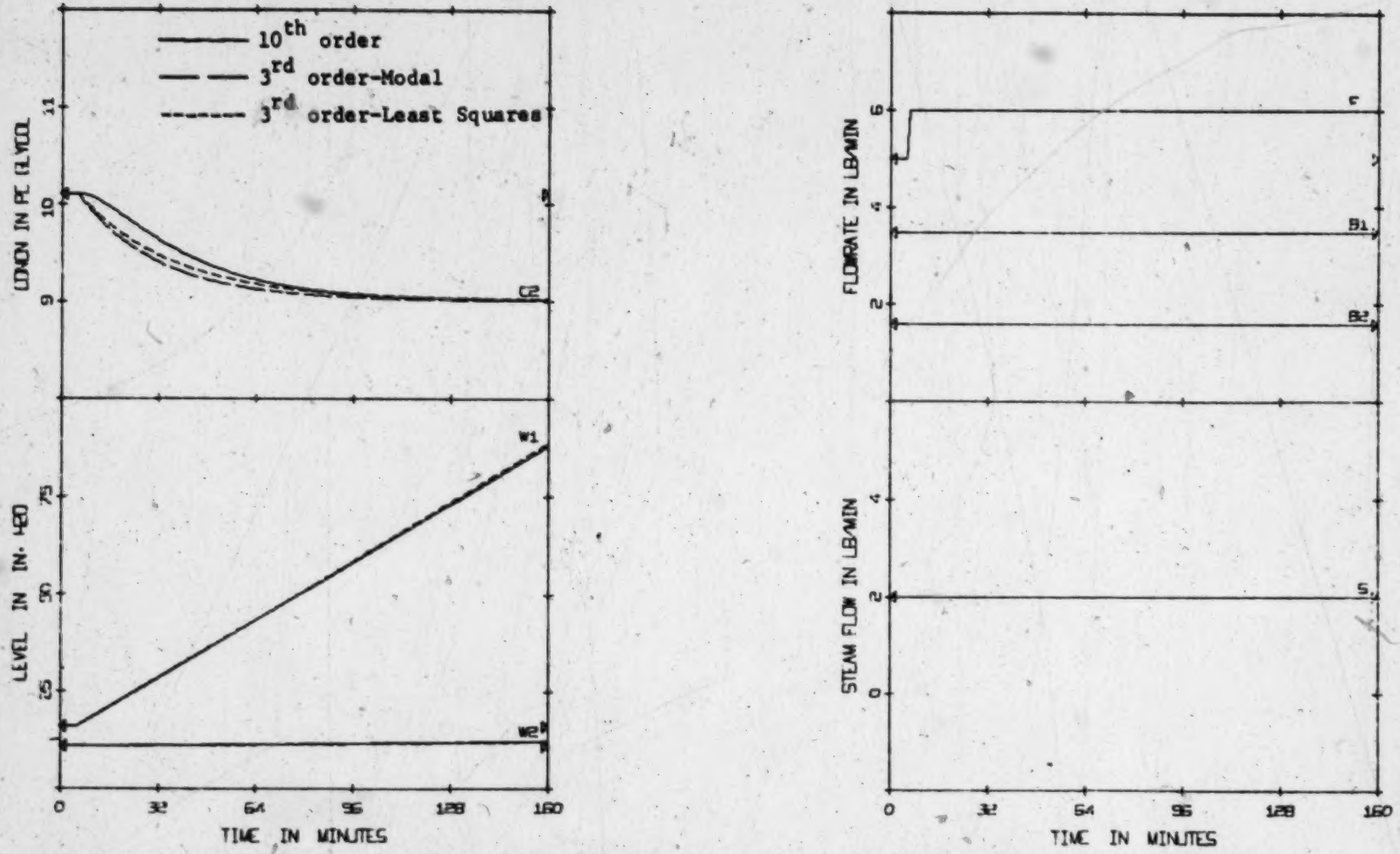


FIGURE 6.6 COMPARISON OF MODELS BY MODAL (MARSHALL'S) APPROACH AND RANDOM LEAST SQUARES APPROACH (10L(2018), 3L(2119), 3L(2050)/D, +20%F/OL/2018, 2119, 2050)

TABLE 6.2

EIGENVALUES OF Φ_R FOR TWO THIRD ORDER MODELS

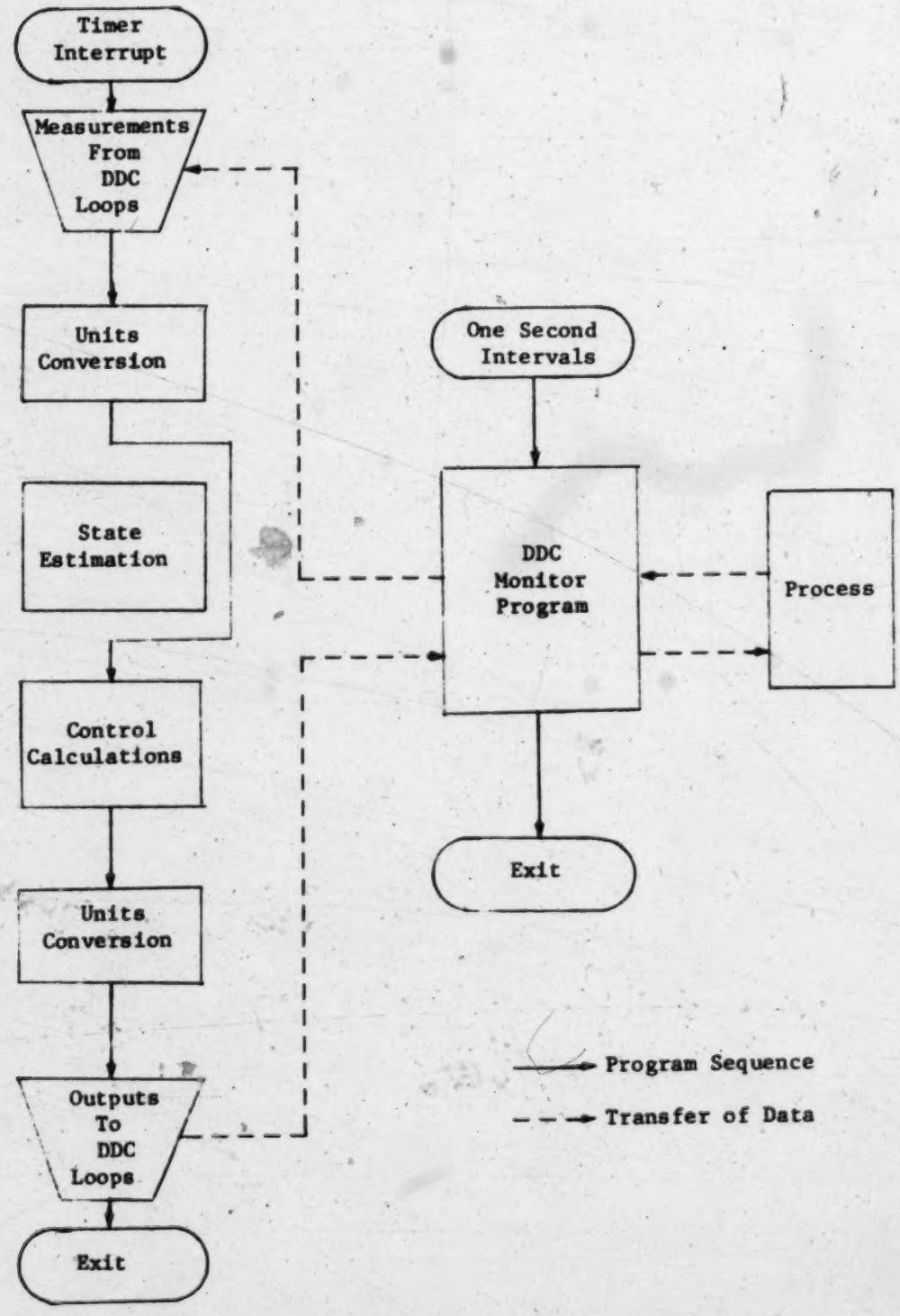
Eigenvalue Number	Marshall's Modal Analysis	Least Squares Analysis
1	1.	1.
2	1.	1.
3	0.9600	0.9662

one) which would tend to exceed physical operating limits if not controlled. The remaining state variables are less critical and their values are not of direct concern.

In the discussion to follow, the control laws will be referred to by an abbreviated code for clarity and brevity. The third order control laws which were designed by reducing the order of the tenth order optimal control law, are referred to as "3RED10" while those which were optimal for a third order model are referred to as "3OPT". Following this designation is reference to the approach used in the model or the control law reduction as either "(M)" if a modal analysis is used or "(LS)" if a least squares analysis is used. Thus, the third order control law reduced from a tenth order controller using the modal analysis is referred to as "3RED10(M)".

The "optimal" control laws mentioned above were calculated using dynamic programming as described by Newell [28]. The weighting matrices in the performance index used with dynamic programming are shown in Appendix A. It should be pointed out here that different weighting matrices could have been used for the third order control laws which would have affected the results. However, to be able to compare the results it was felt that each state should be weighted the same amount in all calculations.

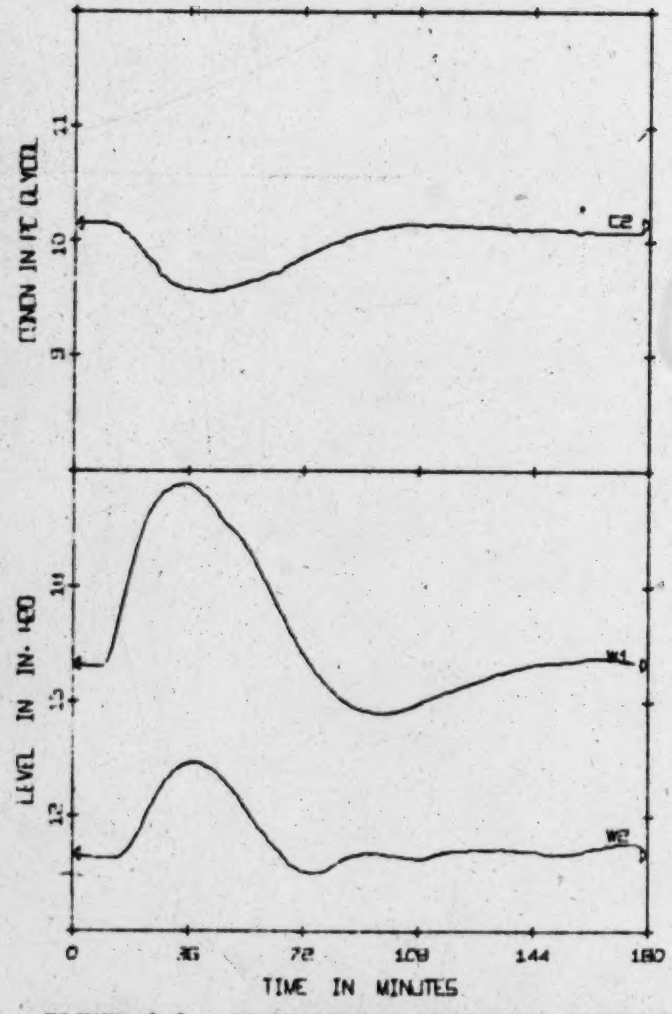
The control laws were applied experimentally for the control of the pilot plant evaporator using the general scheme shown in Figure 6.7. The pilot plant is interfaced to the IBM 1800 computer through the DDC (direct digital control) monitor programs. The



PIGURE 6.7 SCHEMATIC OF MULTIVARIABLE CONTROL PROGRAMS

actual multivariable control programs, written by Newell [29], obtain the process measurements from the DDC loop record table, calculate a set of control values, and then change the setpoints for these control loops in the DDC loop record table. The state estimation step, which has been discussed in detail by Hamilton [22], is not required in this application since all the control laws used are a function of measurable state variables only. The units conversion steps are needed since the control laws require measurements as normalized perturbation variables and calculate a control vector in this same form. A more complete description of the control programs is presented by Newell and Fisher [29] and of the application of these programs to the control of the evaporator by Newell [28].

For comparison with the multivariable control schemes of this chapter, Figure 6.8 shows the response of the evaporator when upset by a +20% increase in feedflow and controlled by a multiloop, proportional plus integral control scheme. Each liquid hold up is controlled by the outlet flow from the particular effect, while the product concentration is controlled by the steam. This was shown by Newell [28] to be the best multiloop configuration. The controller constants and sample time used in this run are shown in Table 6.3 and are identical to the values used by Jacobson [24] for his "standard" case. The sample time of 64 seconds is the same as that used for the multivariable controllers, which will be discussed next. The scales in Figure 6.8 are the same as those in the figures to follow except that the total time is different and the scale of the levels is shifted vertically by two inches.



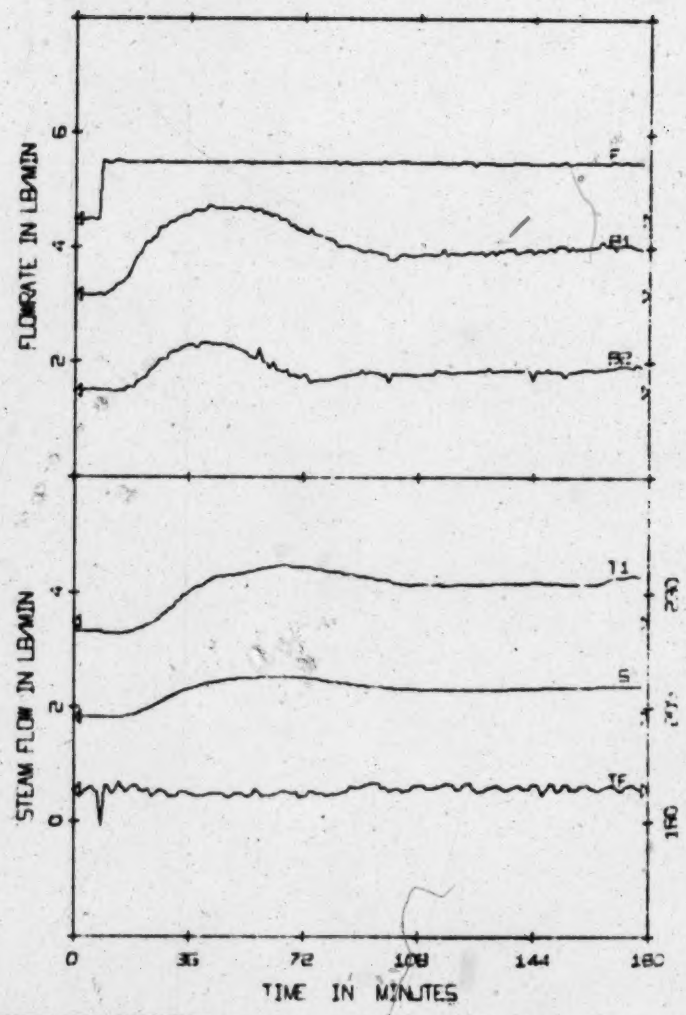


FIGURE 6.8 EXPERIMENTAL MULTILOOP PROPORTIONAL PLUS INTEGRAL FEEDBACK CONTROL (PP/D, +20%F/FB + I/DDC/ML34)

TABLE 6.3

CONTROL LOOP SETTINGS FOR MULTILOOP RUNS

Loop	Control Interval (sec.)	Proportional Constant	Integral Constant (sec.)
B1 W1	64	0.75	2048.
B2 W2	64	2.5	4096.
S C2	64	1.5	2728.

Various multivariable control schemes have been applied to the control of the evaporator. These are discussed in the following sections. For the proportional feedback and the proportional plus integral feedback, control schemes, the results of using a fifth order control law is also presented. This control law is optimal for the fifth order model which was derived by Newell [28]. This will provide a comparison between the work presented in this chapter and the previous work by Newell [28].

6.4.1 Proportional Feedback Plus Feedforward Control

All four third order proportional feedback plus feedforward (FB + FF) control laws which were applied to the evaporator, controlled it well with only slight differences. The control matrices used are shown in Table 6.4, along with the setpoint control matrices to be considered later. The response of the evaporator when controlled by the FB + FF control law designed as

a) 3RED10(M) is shown in Figure (6.9)

b) 3RED10(LS) is shown in Figure (6.10)

c) 3OPT(M) is shown in Figure (6.11)

d) 3OPT(LS) is shown in Figure (6.12).

These responses are all very similar. However, the controllers designed as 3RED10(M) (Figure 6.9) and 3OPT(LS) (Figure 6.12) provide slightly steadier control than the other two, with less oscillatory responses, particularly in the holdups, W1 and W2, and in the controls, B1, B2 and S.

The controller calculated as 3RED10(LS) (Figure 6.10) results in a larger offset in the three controlled variables, W1, W2 and C2.

TABLE 6.4

CONTROL MATRICES FOR PROPORTIONAL FEEDBACK

FEEDFORWARD AND SETPOINT CONTROL

Design Method	$\underline{K}_R^{FB}$	$\underline{K}_R^{FF}$	$\underline{K}_R^{SP}$
3RED10(M)	$\begin{bmatrix} 2.467 & 0.02163 & -4.705 \\ 4.288 & -1.340 & 8.885 \\ 4.128 & 9.760 & 9.528 \end{bmatrix}$	$\begin{bmatrix} 1.238 & -0.5639 & -0.4138 \\ 0.9815 & 0.2177 & -0.00123 \\ 0.9978 & 0.9877 & -0.00140 \end{bmatrix}$	$\begin{bmatrix} -2.468 & -0.0212 & 5.276 \\ -4.289 & 1.340 & -9.100 \\ -4.128 & -9.763 & -10.52 \end{bmatrix}$
3RED10(LS)	$\begin{bmatrix} 8.510 & -0.2903 & -15.32 \\ 2.606 & -1.224 & 11.67 \\ 2.838 & 9.820 & 12.19 \end{bmatrix}$ *	$\begin{bmatrix} 3.314 & -0.2433 & -0.4919 \\ 0.4255 & 0.0787 & 0.02917 \\ 0.8722 & 0.2224 & 0.00975 \end{bmatrix}$	$\begin{bmatrix} -8.711 & -0.0367 & 17.20 \\ -2.569 & 1.345 & -12.20 \\ -2.731 & -9.728 & -13.43 \end{bmatrix}$
30PT(M)	$\begin{bmatrix} 4.904 & -0.4013 & -11.92 \\ 5.784 & -1.600 & 4.425 \\ 4.093 & 9.685 & 9.357 \end{bmatrix}$	$\begin{bmatrix} 1.238 & -0.5583 & -0.4128 \\ 0.9832 & 0.2231 & -0.00055 \\ 0.9983 & 0.9937 & -0.00122 \end{bmatrix}$	$\begin{bmatrix} -4.904 & 0.4017 & 12.47 \\ -5.784 & 1.599 & -4.648 \\ -4.094 & -9.686 & -10.35 \end{bmatrix}$
30PT(LS)	$\begin{bmatrix} 4.914 & -0.2565 & -13.63 \\ 5.786 & -1.654 & 5.182 \\ 4.076 & 9.655 & 9.873 \end{bmatrix}$	$\begin{bmatrix} 1.242 & -0.5561 & -0.4123 \\ 0.9838 & 0.2222 & -0.000574 \\ 0.9943 & 0.9912 & -0.00229 \end{bmatrix}$	$\begin{bmatrix} -4.914 & 0.2564 & 14.19 \\ -5.786 & 1.654 & -5.407 \\ -4.076 & -9.655 & -10.87 \end{bmatrix}$

*This $\underline{K}_R^{FB}$ is used only with the $\underline{K}_R^{FF}$.

The $\underline{K}_R^{FB}$ which is used with $\underline{K}_R^{SP}$ for 3RED10(LS) is

$$\begin{bmatrix} 8.589 & -0.2504 & -15.24 \\ 2.587 & -1.243 & 11.65 \\ 2.799 & 9.825 & 12.17 \end{bmatrix}$$

and that which would be applied for proportional feedback only is

$$\begin{bmatrix} 8.943 & -0.05227 & -15.44 \\ 2.493 & -1.289 & 11.66 \\ 2.647 & 9.804 & 12.20 \end{bmatrix}$$

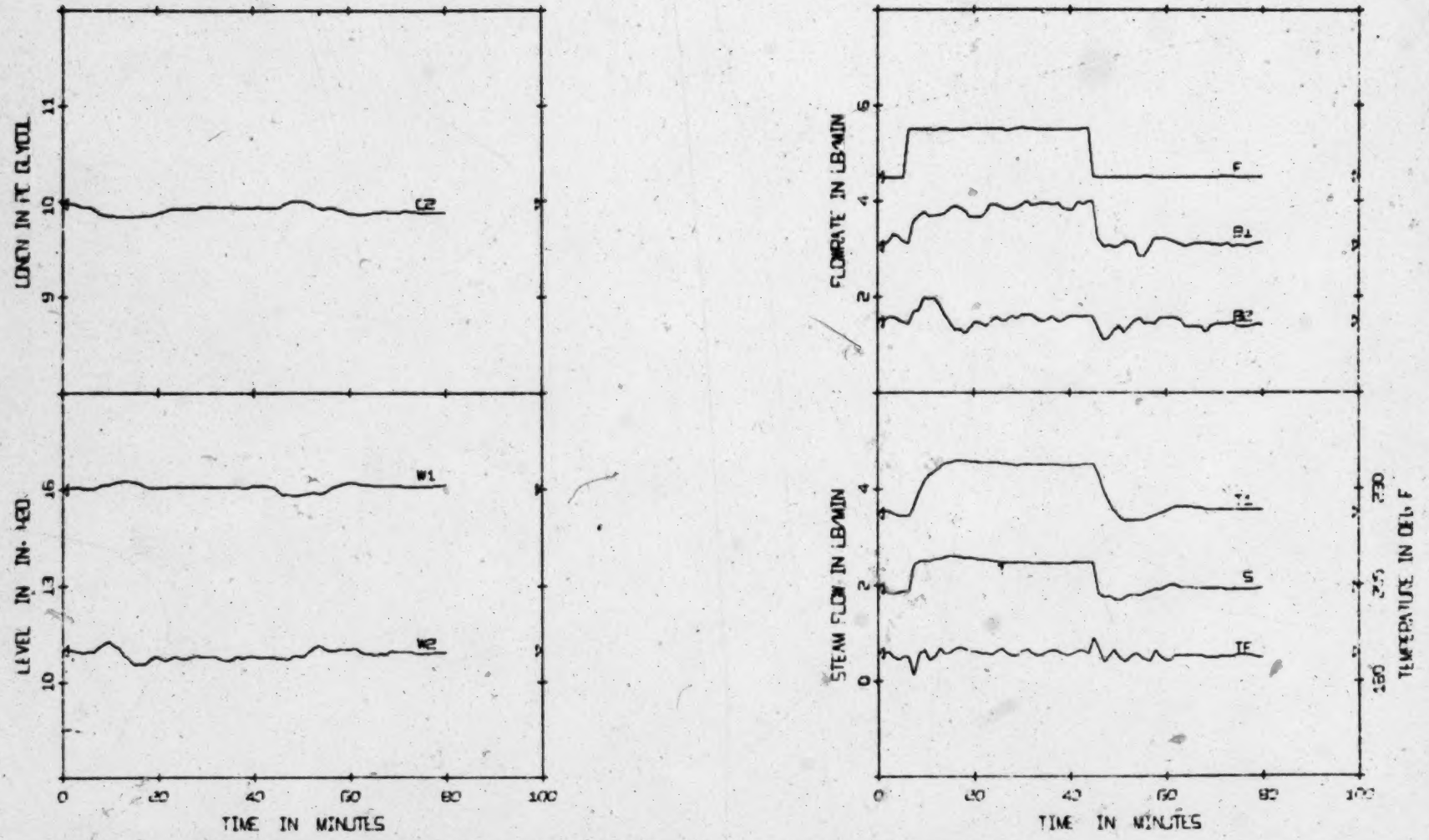


FIGURE 6.9 EXPERIMENTAL PROPORTIONAL FEEDBACK, FEEDFORWARD CONTROL (3RED10(M)) (PP/D, 20%F/FB + FF/3RED10(M)/FF21)

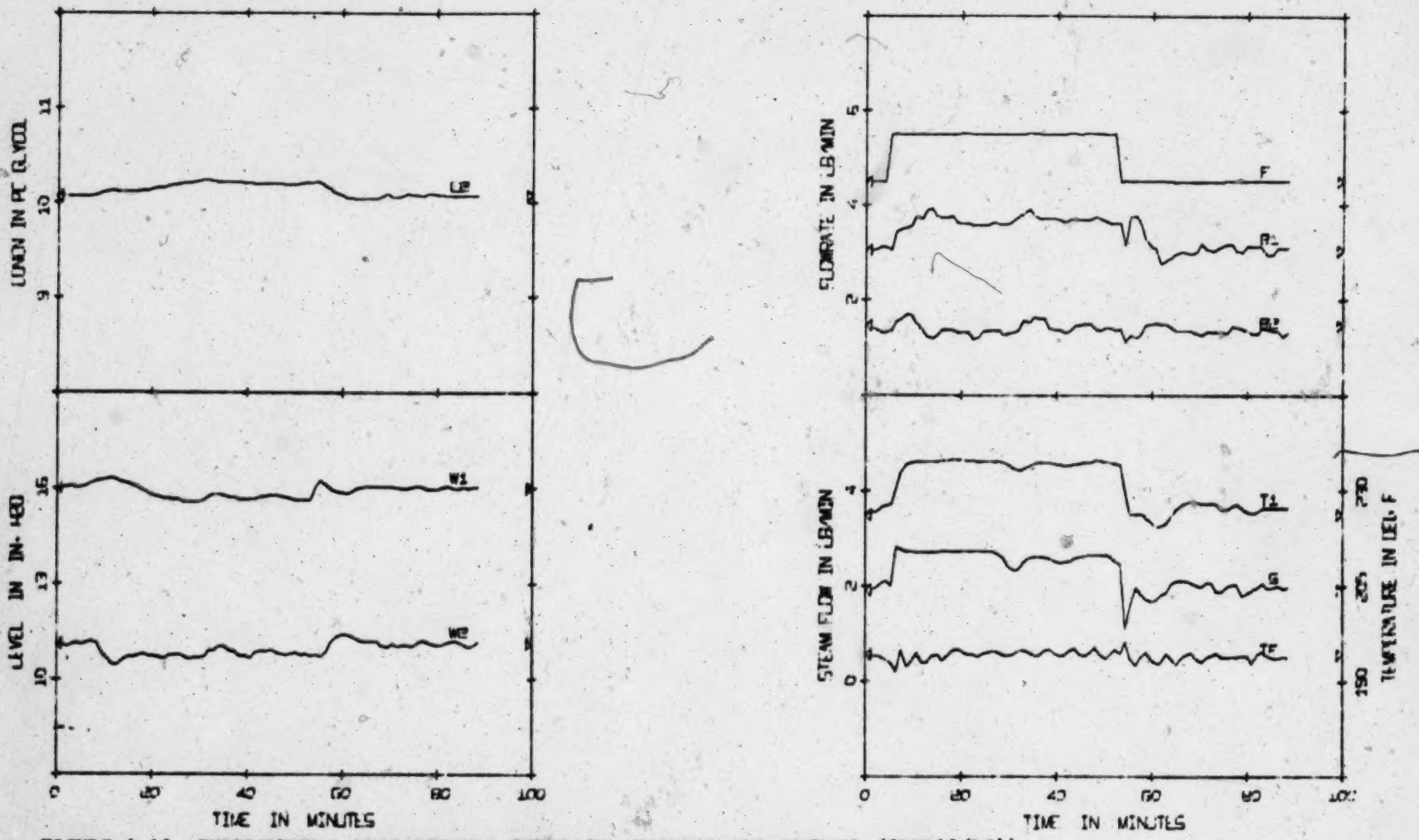


FIGURE 6.10 EXPERIMENTAL PROPORTIONAL FEEDBACK, FEEDFORWARD CONTROL (3RED10(LS)) (PP/D, 20%F/FB + FF/3RED10(LS)/FF23)

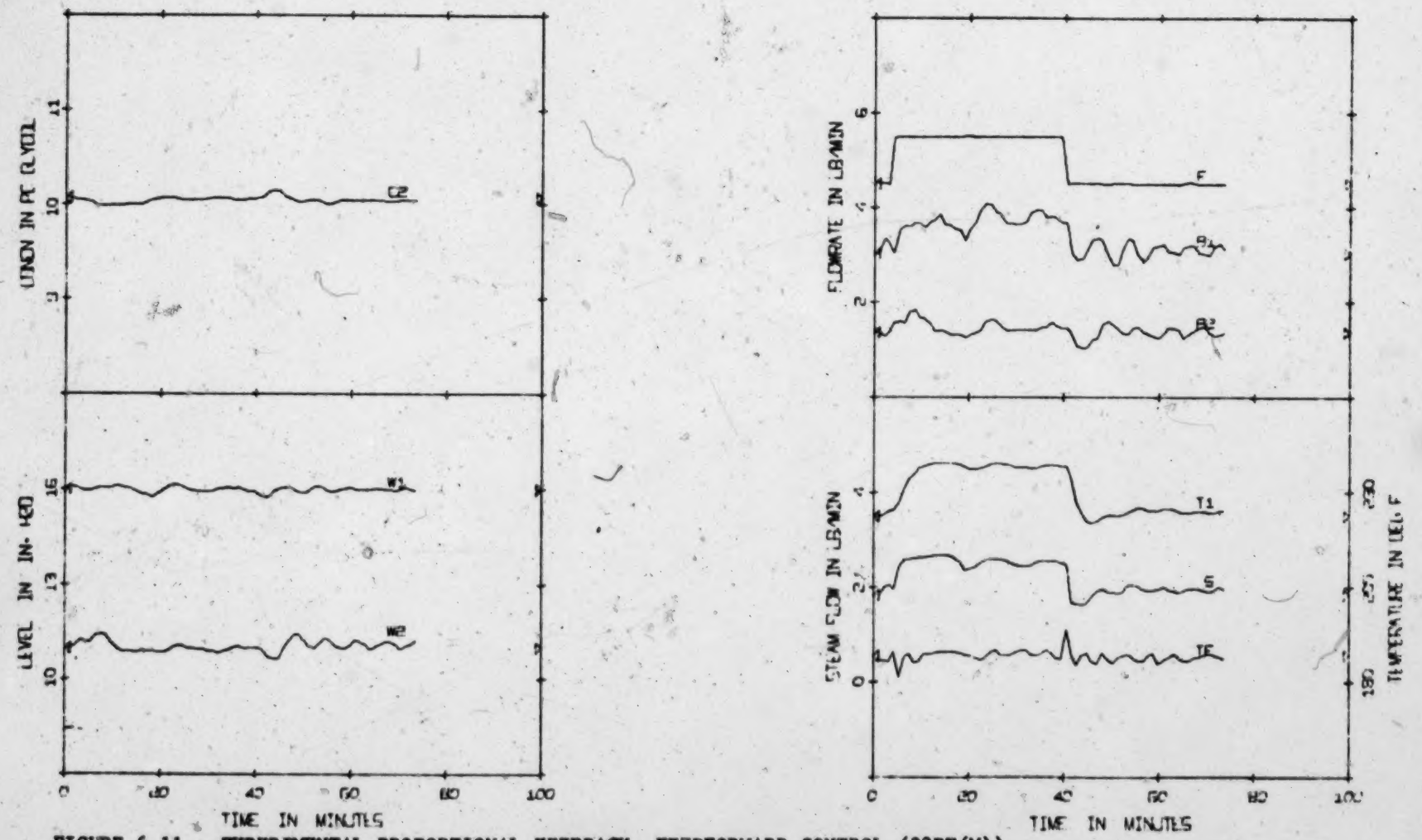


FIGURE 6.11 EXPERIMENTAL PROPORTIONAL FEEDBACK, FEEDFORWARD CONTROL (30PT(M)) (PP/D, 20%F/FB + FF/30PT(M)/FF24)

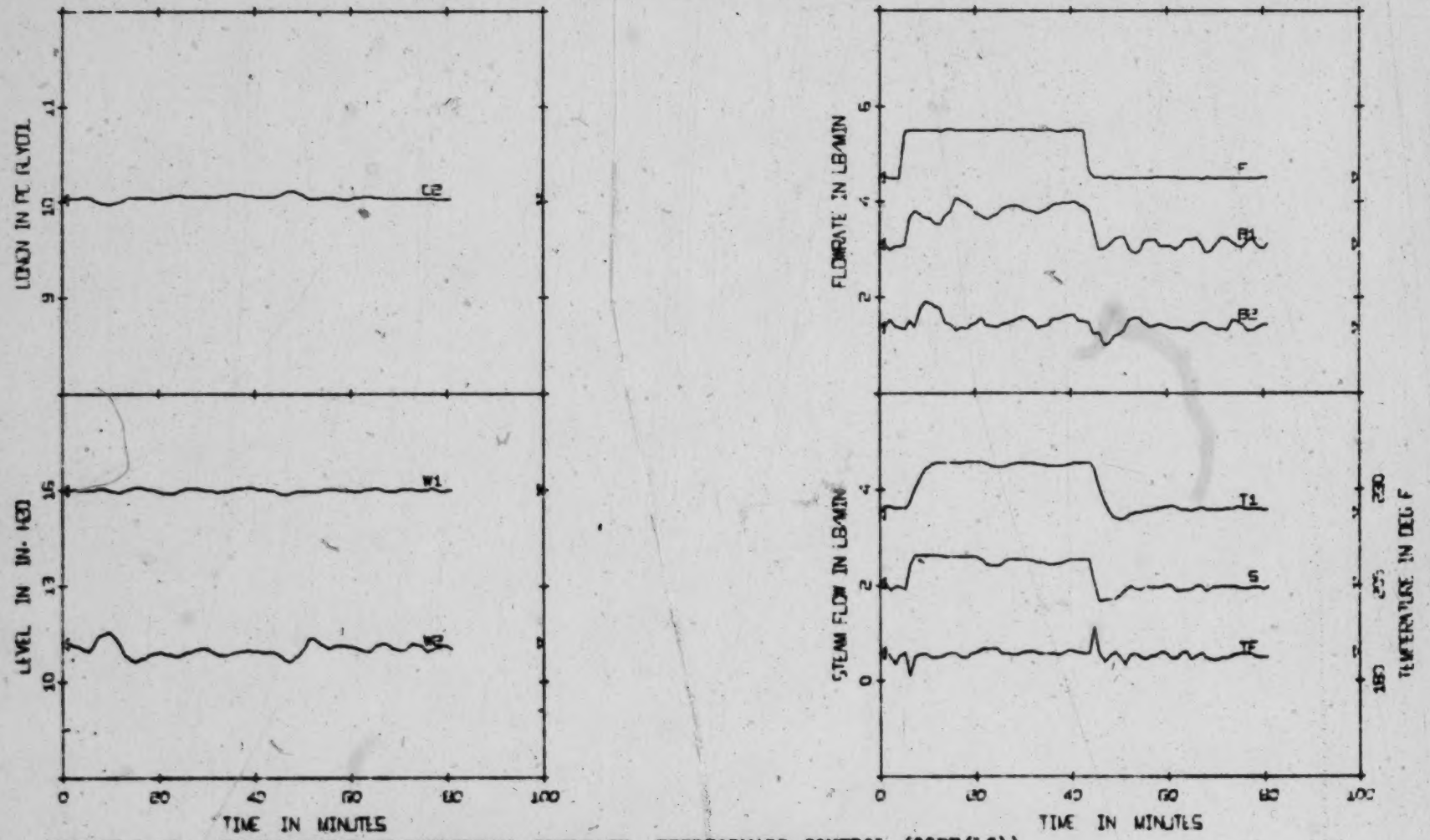


FIGURE 6.12 EXPERIMENTAL PROPORTIONAL FEEDBACK, FEEDFORWARD CONTROL (30PT(LS)) (PP/D, 20XF/FB + FF/30PT(LS)/FF25)

With perfect feedforward control, there should be no steady state offset. This offset results since the control law is calculated with a least squares analysis, with no adjustment for steady states agreement between the system controlled optimally and suboptimally. This offset could be removed by designing $\underline{\underline{K}}_R^{FB}$ alone using least squares, and then designing $\underline{\underline{K}}_R^{FF}$ for zero offsets in the three controlled variables, as was discussed by Newell et al. [30].

Three of the four $\underline{\underline{K}}_R^{FF}$ matrices shown in Table 6.4 are very similar, the only different one is that designed as 3RED10(LS). However, of these three similar $\underline{\underline{K}}_R^{FF}$ matrices, the corresponding $\underline{\underline{K}}_R^{FB}$ matrices are not similar, with some of the larger elements differing by as much as a factor of three.

6.4.2 Proportional Feedback Plus Setpoint Control

The proportional feedback plus setpoint (FB + SP) control laws were applied to the evaporator for two setpoint changes in C2. The first changed C2 from a normalized perturbated value of 0 to +0.1, while the second returned C2 to 0. The form of the control law being used is

$$\underline{u} = \underline{\underline{K}}_R^{FB} \underline{x}_1 + \underline{\underline{K}}_R^{SP} \underline{y}^{SP} \qquad (6.51)$$

Thus, when $\underline{y}^{SP} = \underline{0}$, as it was for the second step (0.1 to 0), the run which resulted was the same as a proportional feedback run with the system initially at a non-zero state.

The response of the evaporator when controlled by the controllers designed as

a) 3RED10(M) is shown in Figure (6.13)

b) 3RED10(LS) is shown in Figure (6.14)

c) 3OPT(M) is shown in Figure (6.15)

d) 3OPT(LS) is shown in Figure (6.16)

All four control systems obtained the desired +10% change in C2. However, the controller designed as 3RED10(M) responded with about half the overshoot (10%) compared with 16% - 19% for the response produced by the other three control laws. Also, the response of the process variables is less oscillatory when controlled by the control law designed as 3RED10(M).

There is a larger difference between the control produced by the four results in the second part of the runs, when $\underline{K}_R^{FB}$ is controlling the process with a non-zero initial condition in C2. All four controllers eventually return the process to its expected steady state. Once again, however, the controller designed as 3RED10(M), (Figure 6.13), controls the system in the smoothest manner without excessive oscillation in any of the process variables. The control calculated as 3RED10(LS) produced a most undesirable response with very large oscillations. This response is not surprising. The $\underline{K}_R^{FB}$ which is being used alone to provide proportional feedback control, was designed along with a setpoint control matrix to provide for changes in the setpoint vector. A better response would result if $\underline{K}_R^{FB}$ had been designed using least squares and then $\underline{K}_R^{SP}$ was calculated to provide for setpoint changes.

The responses for the non-zero initial C2 value of both controllers designed using the third order models are very similar, as

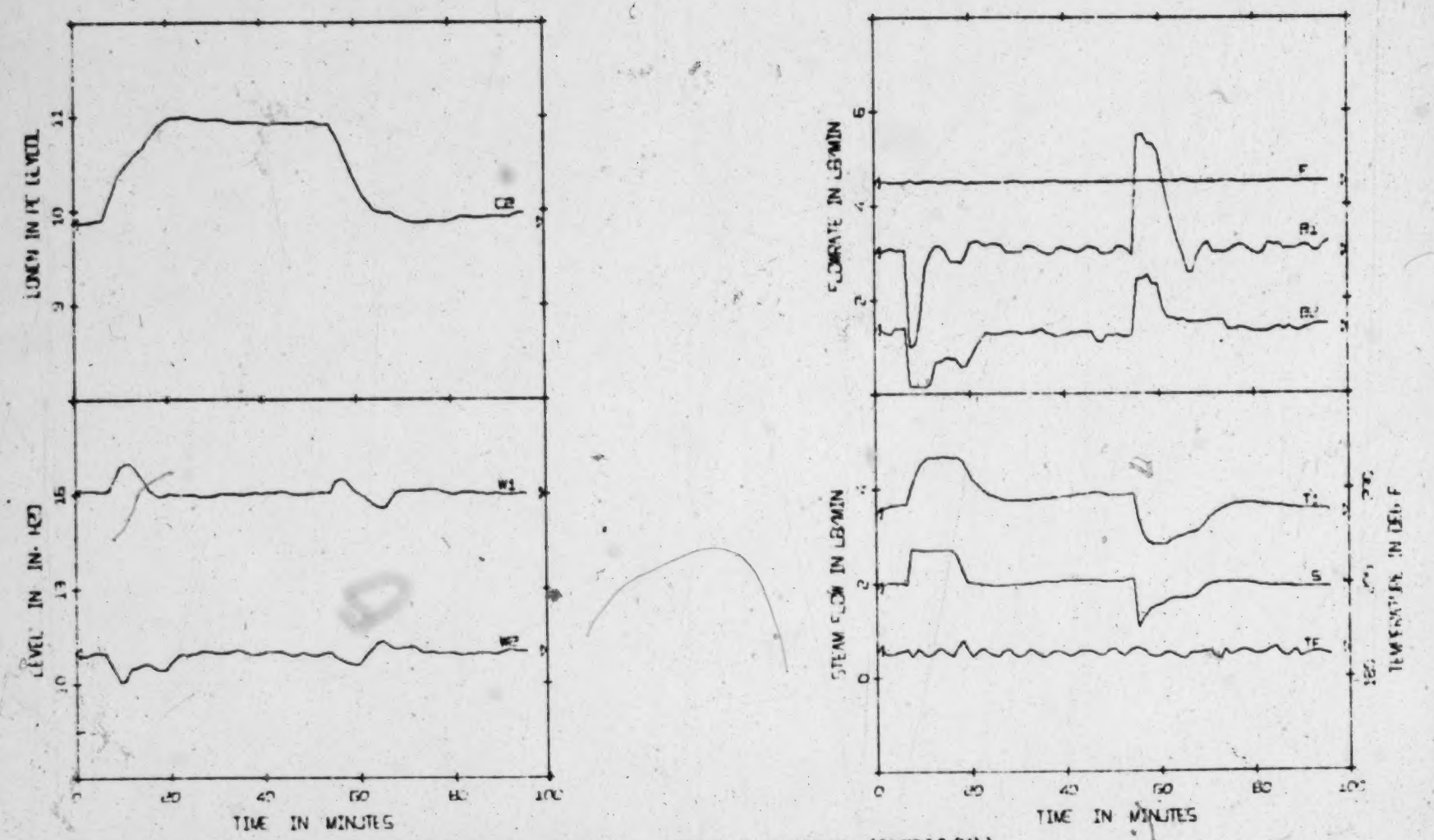
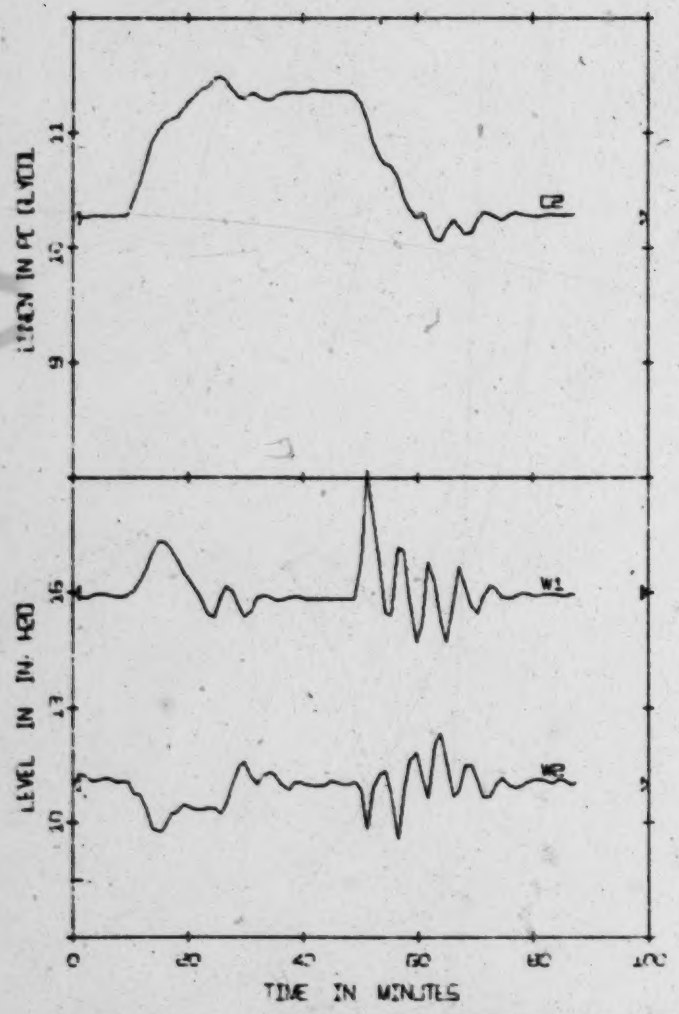


FIGURE 6.13 EXPERIMENTAL PROPORTIONAL FEEDBACK, SETPOINT CONTROL (3RED10(M)) (PP/SP, 10%C2/FB + SP/3RED10(M)/SP22)



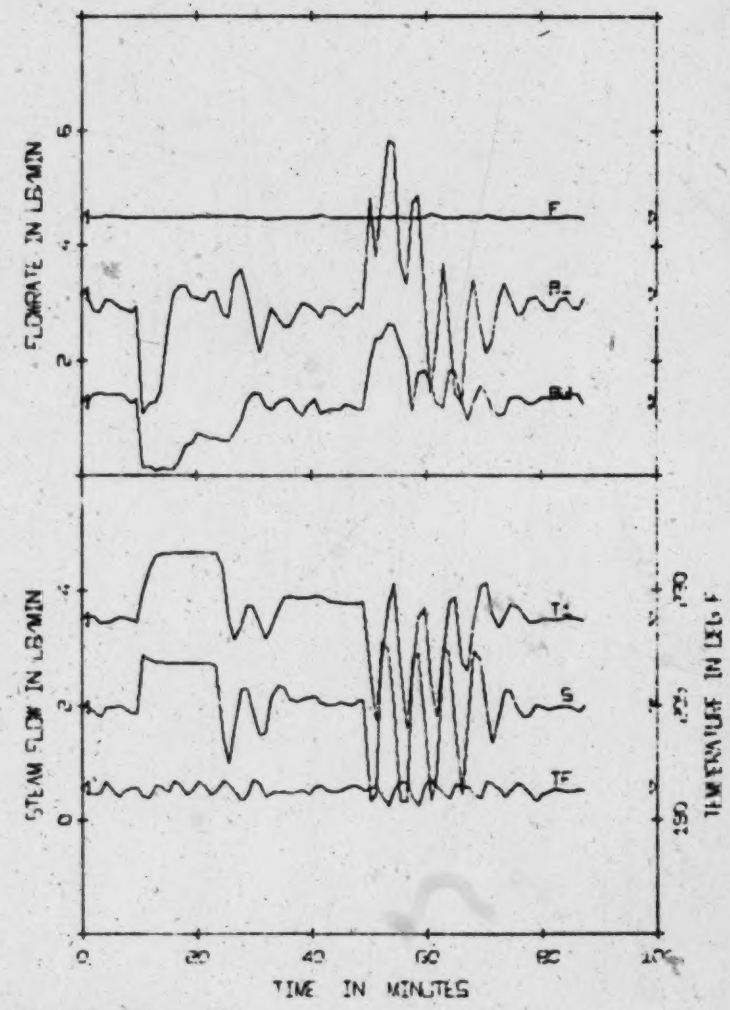


FIGURE 6.14 EXPERIMENTAL PROPORTIONAL FEEDBACK, SETPOINT CONTROL (3RED10(LS)) (PP/SP, 10%C2/FB + SP/3RED10(LS)/SP28)

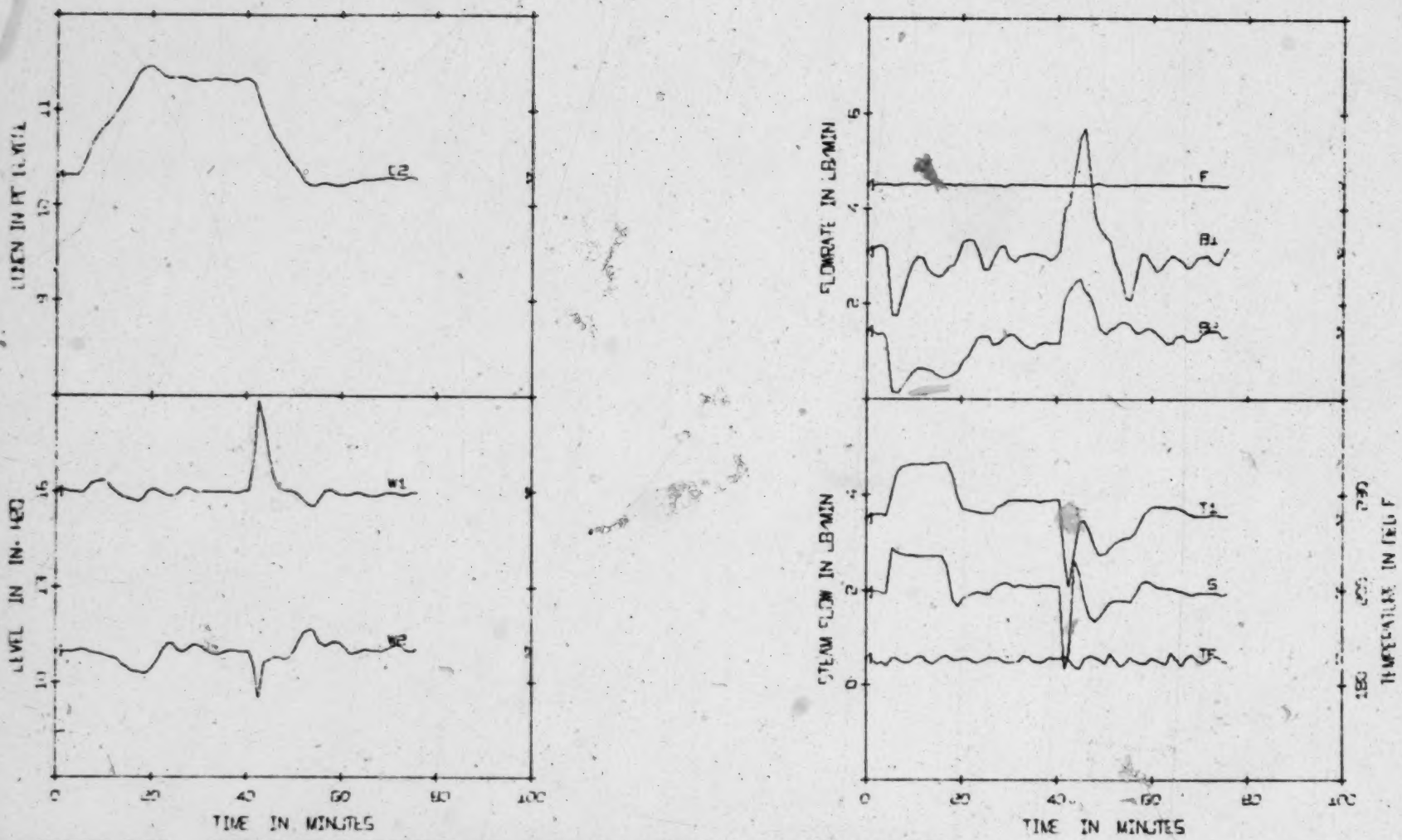


FIGURE 6.15 EXPERIMENTAL PROPORTIONAL FEEDBACK, SETPOINT CONTROL (30PT(M))
(PP/SP, 10%C2/FB + SP/30PT(M)/SP26)

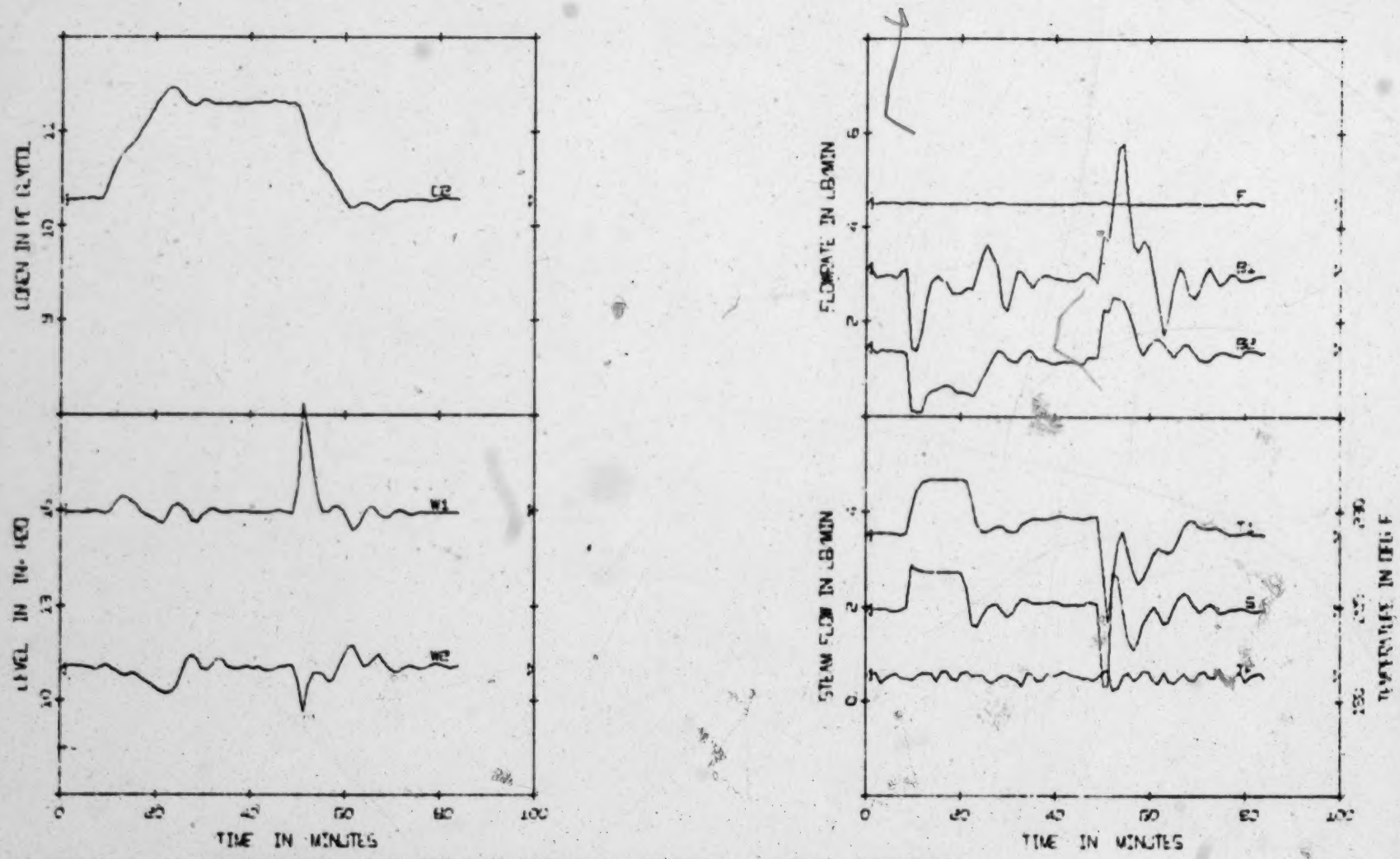


FIGURE 6.16 EXPERIMENTAL PROPORTIONAL FEEDBACK, SETPOINT CONTROL (30PT(LS)) (PP/SP, 10%C2/FB + SP/30PT(LS)/SP27)

shown in Figures 6.15 and 6.16. Both allow the levels to depart by a large amount from their initial (and desired) values. Both responses are more oscillatory than that produced by the control law designed as 3RED10(M).

The control matrices in these FB + SP runs are those shown in Table 6.4. The two $\underline{K}_R^{SP}$ matrices, that were optimal for the two reduced order models, are very similar. The other two $\underline{K}_R^{SP}$ matrices are considerably different, with some large elements differing by a factor of three. Each of these $\underline{K}_R^{SP}$ matrices are used with the same $\underline{K}_R^{FB}$ as was used with the $\underline{K}_R^{FF}$ except for the controller designed as 3RED10(LS).

6.4.3 Proportional Plus Integral Feedback Control

Only three reduced order proportional plus integral (FB + I) control laws were applied to the control of the pilot plant, and of these, only one resulted in satisfactory control. The FB + I control matrices that were calculated are shown in Table 6.5.

The only reduced order FB + I control law which successfully controlled the evaporator was designed as 3RED10(M). This run is shown in Figure 6.17. The controller returned each of the controlled variables W1, W2, and C2 to their desired steady state values.

The response of the evaporator being controlled by the control law designed as 30PT(M) and 30PT(LS) is shown in Figures 6.18 and 6.19, respectively. For each of these runs, only the +20% change in feedflow was introduced since these runs were tending to go un-

TABLE 6.5

CONTROL MATRICES FOR PROPORTIONAL-PLUS-INTEGRAL CONTROL

Design Method	$\underline{K}_R^{FB}$	$\underline{K}_R^{I}$
3RED10(M)	$\begin{bmatrix} 2.977 & 0.0937 & 5.408 \\ 5.066 & -1.246 & 8.10 \\ 5.686 & 12.29 & 11.58 \end{bmatrix}$	$\begin{bmatrix} 0.4117 & 0.02088 & -0.4861 \\ 0.8398 & -0.2887 & 0.8159 \\ 0.8559 & 1.945 & 1.068 \end{bmatrix}$
3RED10(LS)*	$\begin{bmatrix} 11.93 & 0.2050 & -20.14 \\ 3.173 & -1.251 & 11.07 \\ 3.959 & 12.26 & 14.74 \end{bmatrix}$	$\begin{bmatrix} 1.607 & -0.01910 & -1.940 \\ 0.5625 & -0.2608 & 1.096 \\ 0.6176 & 1.953 & 1.398 \end{bmatrix}$
30PT(M)	$\begin{bmatrix} 5.49 & -0.1903 & -12.00 \\ 6.429 & -1.386 & 4.487 \\ 5.519 & 12.26 & 11.81 \end{bmatrix}$	$\begin{bmatrix} 0.9893 & -0.05066 & -1.175 \\ 1.156 & -0.3255 & 0.4373 \\ 0.8254 & 1.935 & 1.090 \end{bmatrix}$
30PT(LS)	$\begin{bmatrix} 5.510 & -0.03945 & -13.80 \\ 6.429 & -1.440 & 5.248 \\ 5.496 & 12.22 & 12.48 \end{bmatrix}$	$\begin{bmatrix} 0.9921 & -0.01645 & -1.337 \\ 1.156 & -0.3384 & 0.5074 \\ 0.8216 & 1.926 & 1.144 \end{bmatrix}$

* This case was not applied to the evaporator due to an unstable simulated response.

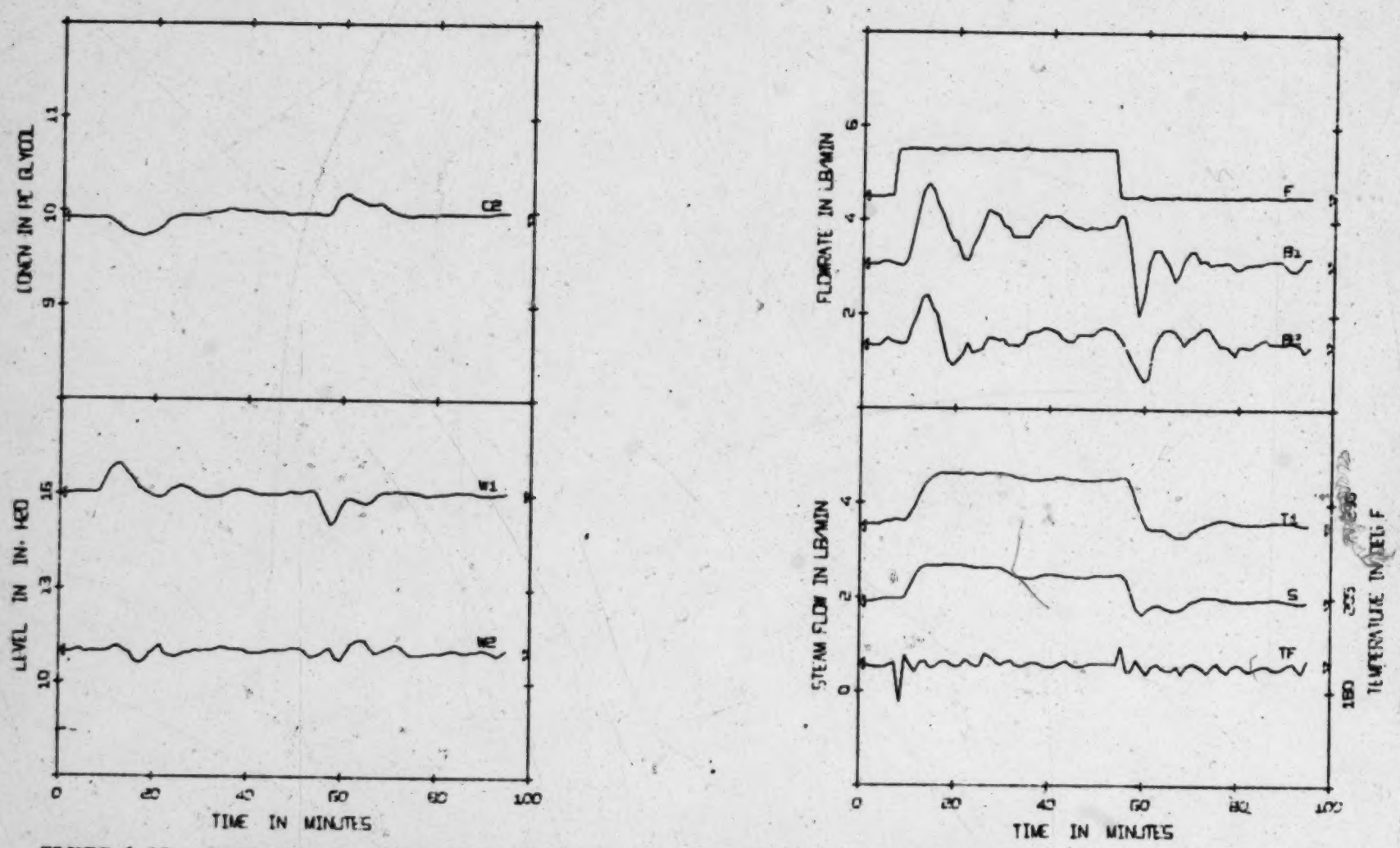


FIGURE 6.17 EXPERIMENTAL PROPORTIONAL PLUS INTEGRAL FEEDBACK CONTROL (3RED10(M)) (PP/D, 20%F/FB + I/3RED10(M)/FB20)

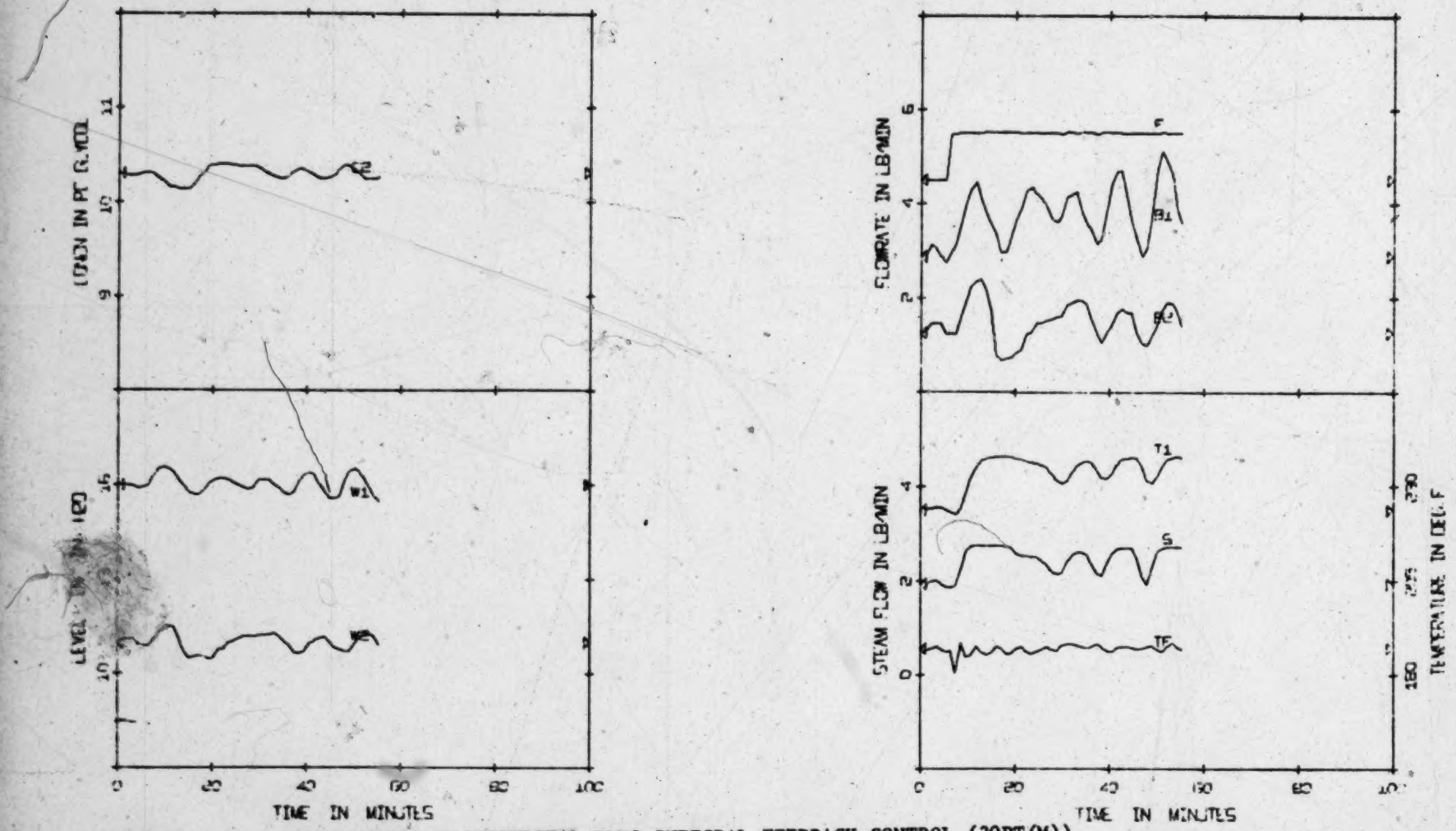


FIGURE 6.18 EXPERIMENTAL PROPORTIONAL PLUS INTEGRAL FEEDBACK CONTROL (30PT(M))
(PP/D, +20%F/FB + I/30PT(M)/FB29)

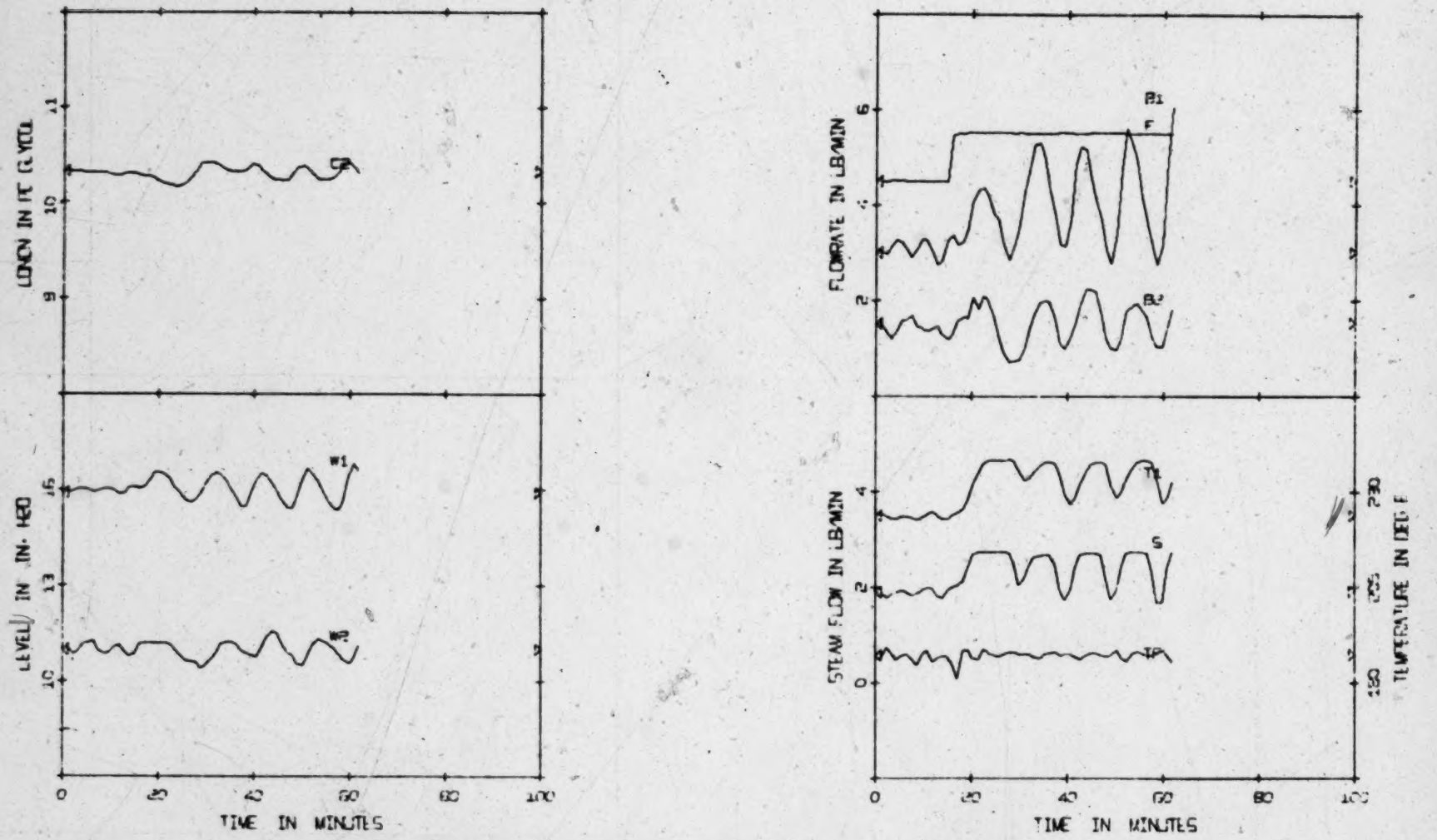


FIGURE 6.19 EXPERIMENTAL PROPORTIONAL PLUS INTEGRAL FEEDBACK CONTROL (30PT(LS)) (PP/D, +20%F/FB + I/30PT(LS)/FB31)

stable as shown by the oscillations in all the controlled and control variables. The control matrices for these two approaches are nearly identical, as shown in Table 6.5.

The FB + I control law designed as 3RED10(LS) was not applied to the pilot plant since its response was unstable in the simulation. A comparison of these matrices with the other matrices in Table 6.5 shows the elements in the result designed as 3RED10(LS) are considerably larger than the corresponding elements in the other matrices. These larger elements cause the oscillatory unstable response. A comparison of the matrices designed as 3RED10(LS) with the FB + I matrices of the tenth order control law in Table A.5 is also interesting. $\underline{K}_R^I$ is almost the same as $\underline{K}^I$. The columns of $\underline{K}_R^{FB}$ are nearly the same as the corresponding columns of $\underline{K}^{FB}$ (columns 3, 7 and 8). Thus, the approach 3RED10(LS) has produced a result which is essentially the same as the tenth order control law with the undesired columns of $\underline{K}^{FB}$ set to zero. Considering this, the unstable simulated response is not surprising.

For purpose of comparison, the control of the evaporator by a fifth order FB + I control law which was optimal for a fifth order model is shown in Figure 6.20. For this control law, $\underline{x}_1^T$ = [W1, C1, H1, W2, C2]. In the application of this control law the state estimation procedure used by Newell [28] provided an estimate of the unmeasurable state variable, C1. The response shown in Figure 6.20 is slightly more oscillatory, but with smaller maximum deviations in C2 and W1 than the responses shown in Figure 6.17

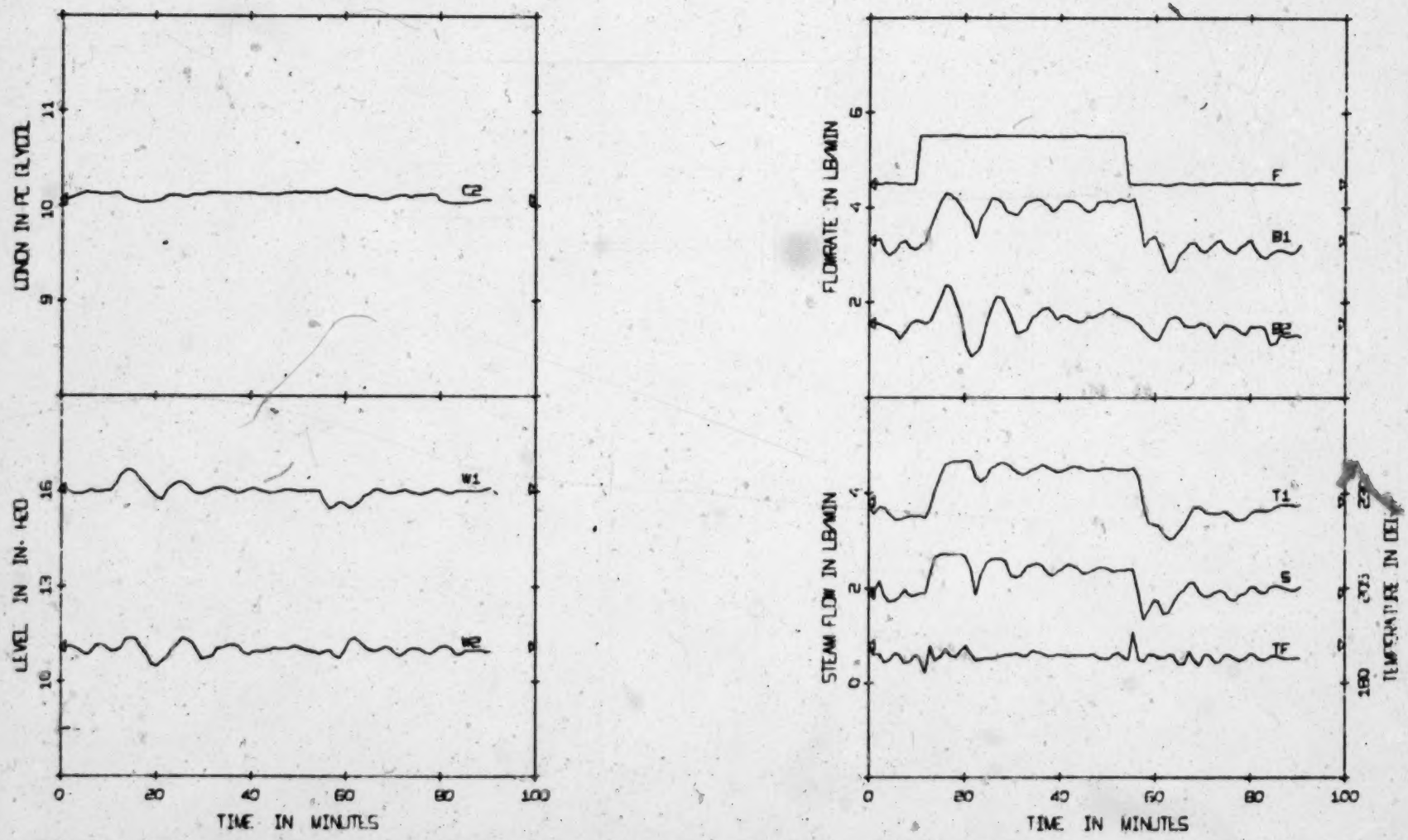


FIGURE 6.20 EXPERIMENTAL FIFTH ORDER PROPORTIONAL PLUS INTEGRAL FEEDBACK CONTROL (PP/D, 20%F/FB + I/50PT/FB18)

(control law designed as 3RED10(M)). Thus, the control provided by the third order control law is not much poorer than that provided by this fifth order controller. The fifth order FB + I control matrices used for this comparison are shown in Table 6.6

6.4.4 Proportional Feedback Control

To complete the comparison of the control laws obtained by a modal analysis, the proportional feedback (FB) controllers were applied as calculated by 3RED10(M) and 30PT(M). These are shown in Figures 6.21 and 6.22, respectively. The control matrices used here are the $\underline{K}_R^{FB}$ shown in Table 6.4. The response shown in Figure 6.21 results in the expected offsets after the first change in feed flow rate. These offsets were eliminated, as expected, after the feed flow was returned to its original value. All the controlled and control variables react very smoothly, without oscillations. The response shown in Figure 6.22 is more oscillatory than that in Figure 6.21. However, after the first step in feed flow there is a smaller offset than that shown in Figure 6.21. This is because this proportional controller (30PT(M)) has generally higher gains and so has overcorrected causing the oscillatory response and the smaller offset.

The feedback controller designed as 3RED10(M) was also used to recover the two oscillatory FB + I control runs shown in Figures 6.18 and 6.19. These "recovery" runs are shown in Figures 6.23 and 6.24. In each case, after the oscillatory FB + I run was stopped and this FB controller started, the evaporator responses settled out immediately. This demonstrates the use of the 3RED10(M) FB controller

TABLE 6.6

CONTROL MATRICES FOR FIFTH ORDER

PROPORTIONAL PLUS INTEGRAL FEEDBACK CONTROL

$$\underline{K}^{FB} = \begin{bmatrix} 8.209 & -1.239 & -3.640 & 0.1391 & -15.44 \\ 4.542 & 0.3714 & 0.5527 & -1.284 & 9.070 \\ 4.238 & 1.166 & -0.05915 & 12.26 & 14.22 \end{bmatrix}$$

$$\underline{K}^{I} = \begin{bmatrix} 1.277 & 0.02747 & -1.432 \\ 0.7931 & -0.2981 & 0.8948 \\ 0.6475 & 1.936 & 1.306 \end{bmatrix}$$

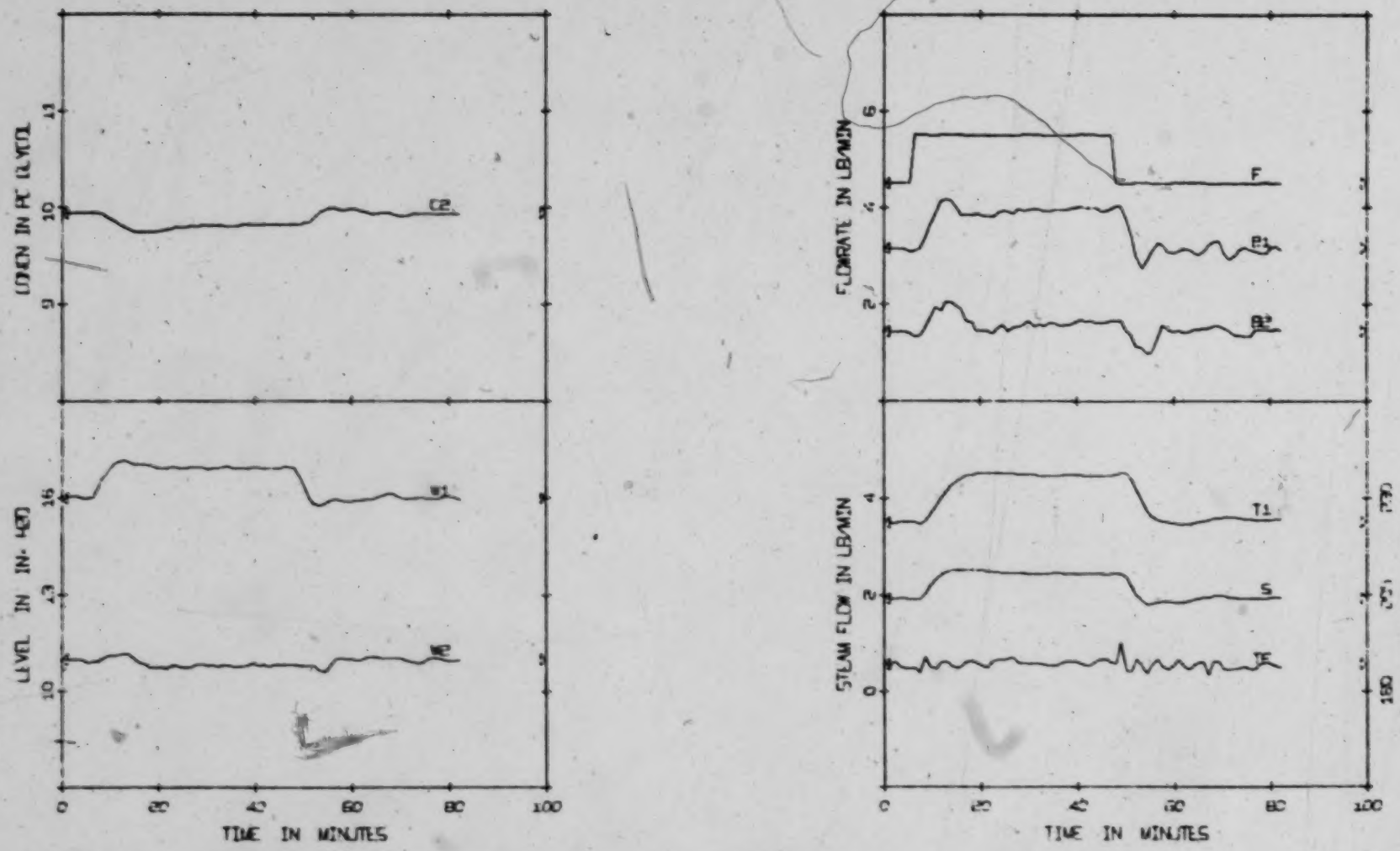
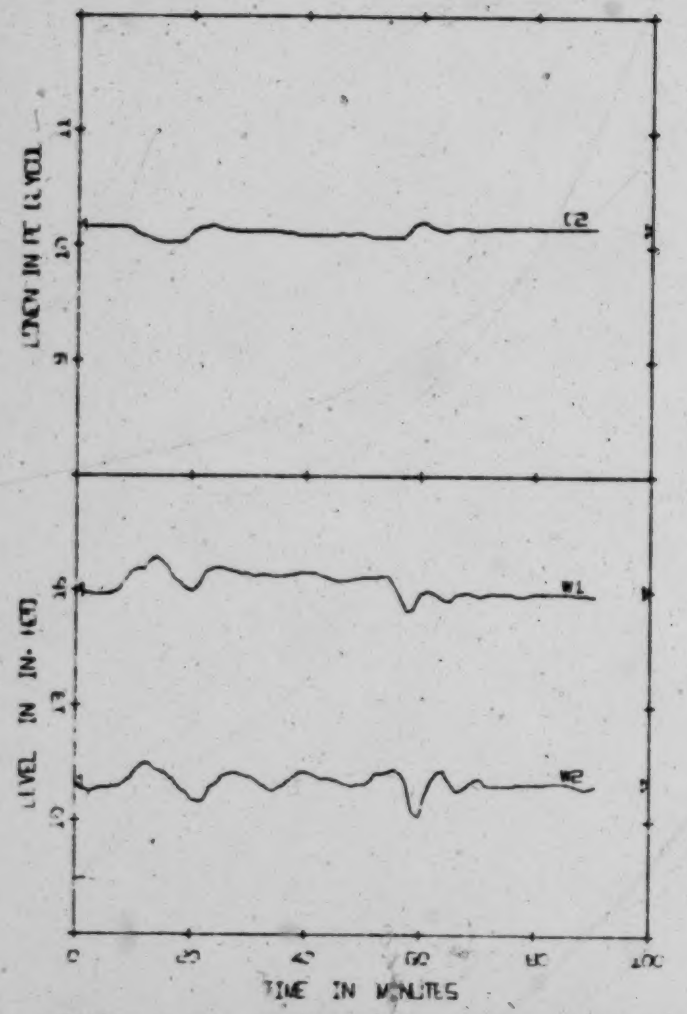


FIGURE 6.21 EXPERIMENTAL PROPORTIONAL FEEDBACK CONTROL (3RED10(M)) (PP/D, 20%F/FB/3RED10(M)/FB19)



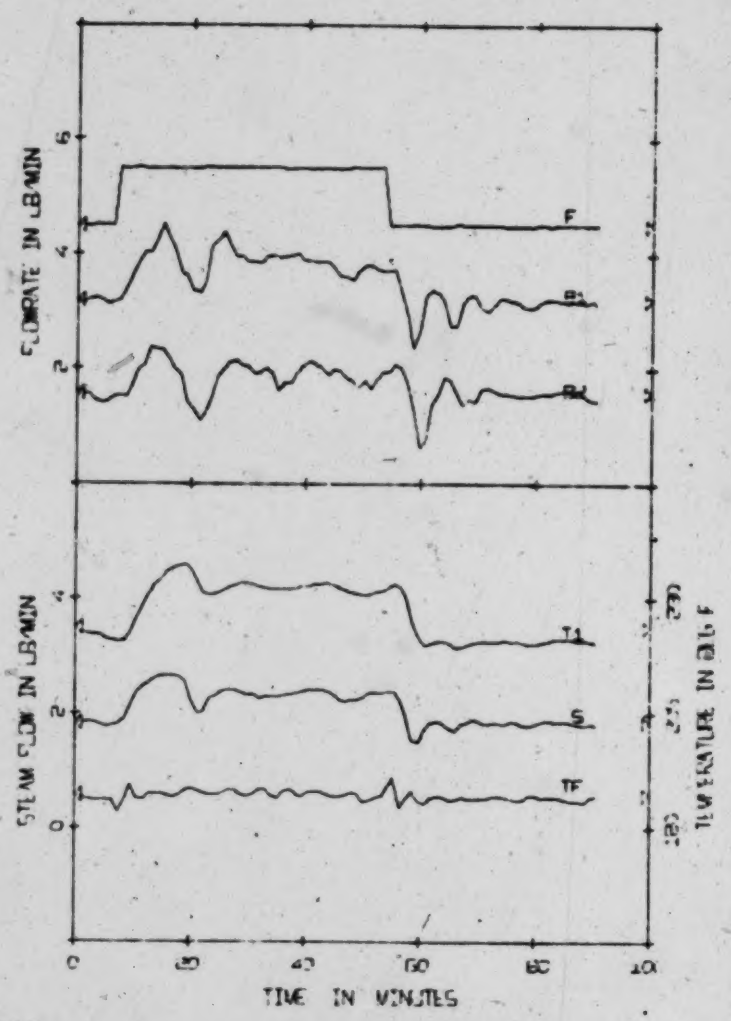


FIGURE 6.22 EXPERIMENTAL PROPORTIONAL FEEDBACK CONTROL (3OPT(M)) (PP/D, 20%F/FB/3OPT(M)/FB33)

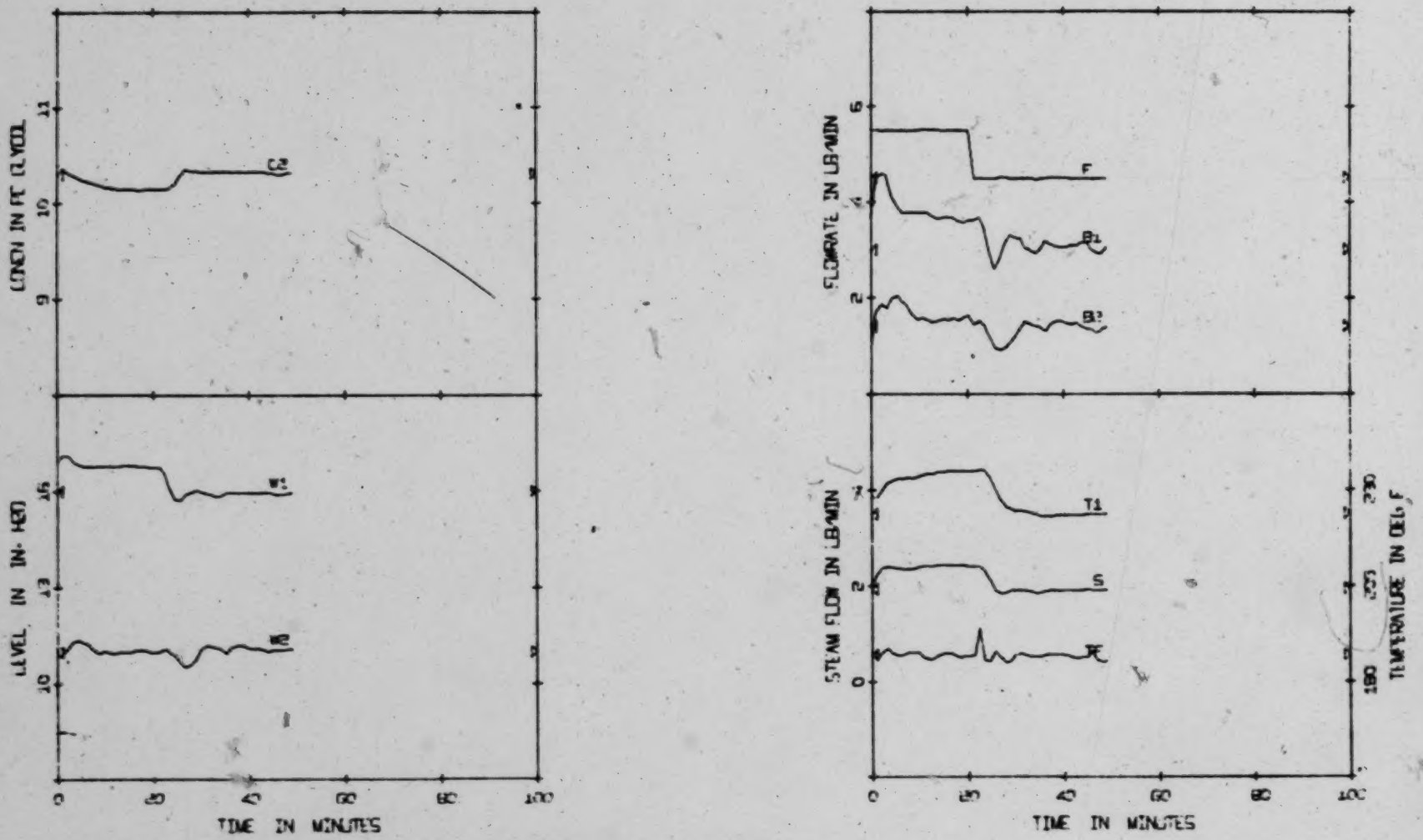
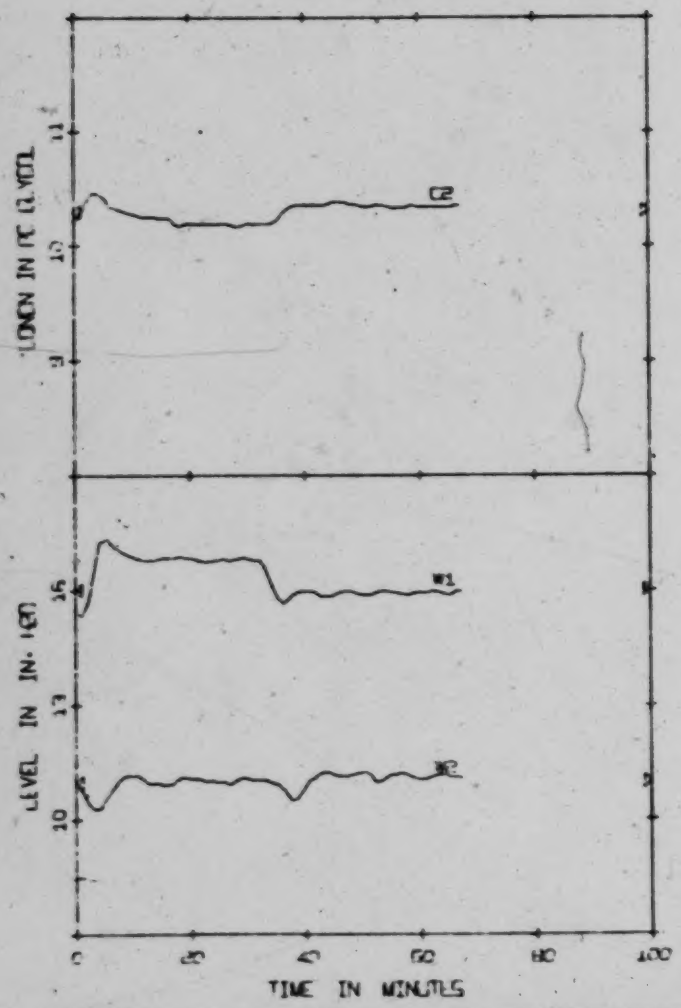


FIGURE 6.23 RECOVERY OF RUN SHOWN IN FIGURE 6.19 (3RED10(M)) ($PP/\underline{x}(0) \neq \underline{0}$, D-20ZF/FB/3RED10(M)/FB30)



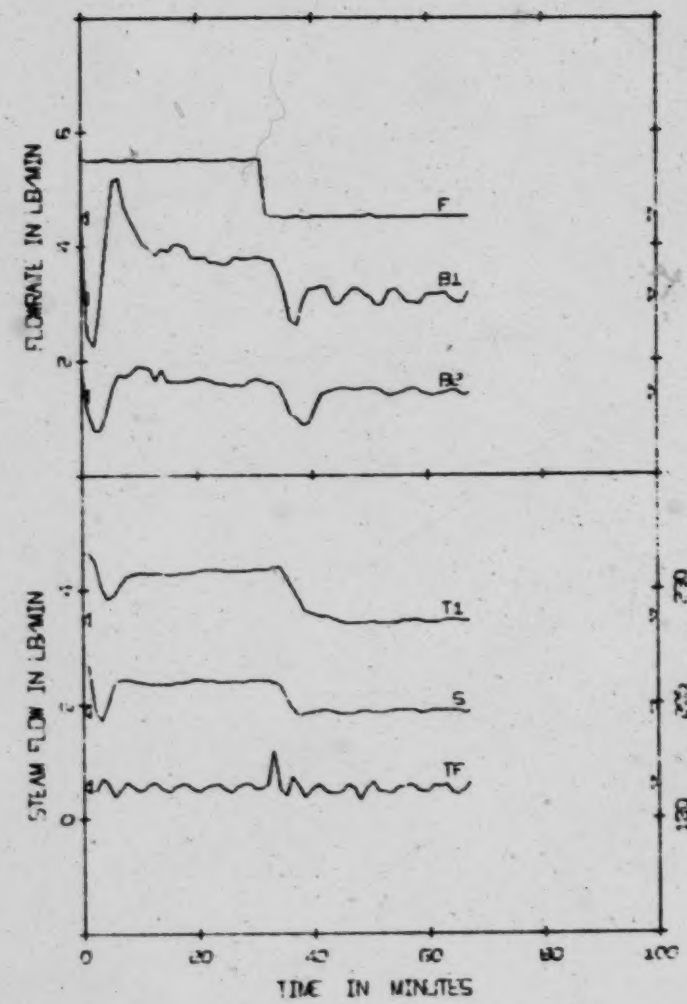


FIGURE 6.24 RECOVERY OF RUN SHOWN IN FIGURE 6.20 (3RED10(M)) ($PP/\underline{x}(0) \neq \underline{0}$, D-20%F/FB/3RED10(M)/FB32)

in controlling a process at an initial non-zero state. The response of the evaporator to the -20% feedflow change in Figures 6.23 and 6.24 is similar to that for the -20% feedflow change in Figure 6.21.

The response of the evaporator controlled with the fifth order controller as 50PT is shown in Figure 6.25. It shows smaller maximum deviations than the responses controlled by the third order controller with smooth controller action in B1, B2 and S. Thus, the third order FB control law provides only slightly poorer control than a fifth order controller. The fifth order FB matrix used for this comparison is shown in Table 6.7.

6.4.5 Summary of Experimental Results

The design approach which produced the most consistent, stable, smooth acting reduced order controllers of all the types considered was that which obtained the third order control law by reducing the order of the tenth order controller using the modal analysis, (3RED10(M)). This is especially evident for the case of proportional plus integral feedback control where it is the only reduced order controller to produce a stable experimental response. This design approach also produced control laws which resulted in nearly as good control as did the fifth order control laws designed from a fifth order model and applied to the evaporator by Newell [28].

Both the third order models, which were used to calculate the control laws as 30PT, were very similar. This accounts for the similarity of the resulting reduced order controllers and of the resulting controlled responses. The simulated comparison of pro-

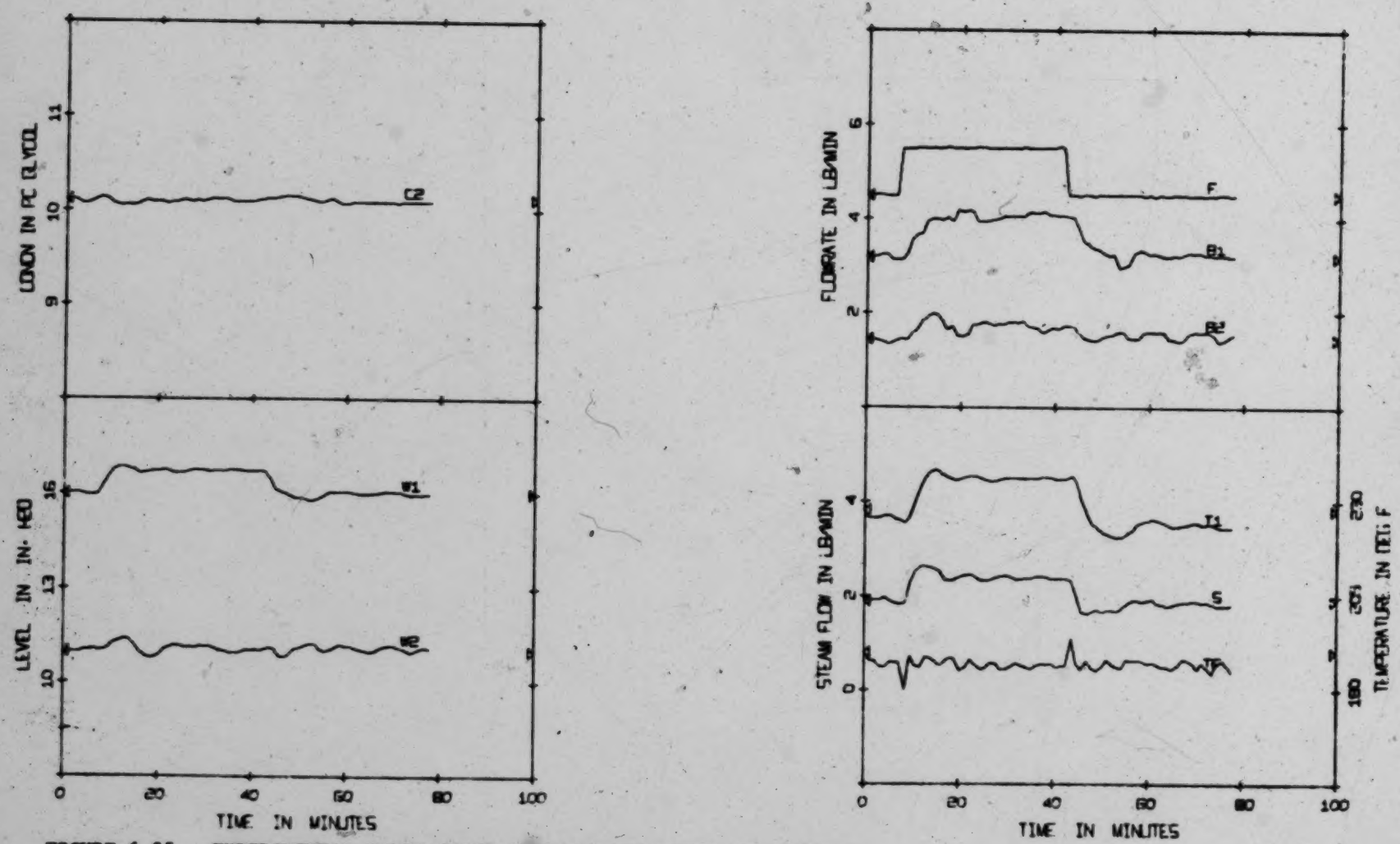


FIGURE 6.25 EXPERIMENTAL FIFTH ORDER PROPORTIONAL FEEDBACK CONTROL (PP/D, 20%F/FB/50PT/FB17)

TABLE 6.7

CONTROL MATRIX FOR FIFTH ORDER

PROPORTIONAL FEEDBACK CONTROL

$$\underline{K}^{FB} = \begin{bmatrix} 6.508 & -1.229 & -3.235 & -0.08719 & -13.14 \\ 3.825 & 0.3688 & 0.6885 & -1.378 & 9.708 \\ 3.058 & 1.056 & 0.1784 & 9.750 & 11.56 \end{bmatrix}$$

portional feedback control obtained from these third order models, was also almost identical.

The approach taken in reducing the high order control laws with a least squares analysis was unsatisfactory as is shown by the results using this method. Also, a different $\underline{K}_R^{FB}$ must be used in each of the four types of control considered. This is undesirable. If a process were being controlled by a feedback feedforward scheme and a setpoint change must be entered, the entire control law must be changed, the new setpoint entered and the feedback feedforward control law put back after the new setpoint has been reached. This control law swapping would not be necessary if the same $\underline{K}_R^{FB}$ could be used for all control types. Thus, an alternate design approach would be to design $\underline{K}_R^{FB}$ using least squares and to design the other matrices separately. The matrix $\underline{K}_R^{FF}$ can be designed to compensate for measurable disturbances as was discussed by Newell et al. [30], $\underline{K}_R^{SP}$ to provide for setpoint control, and $\underline{K}_R^{I}$ to eliminate offsets caused by unmeasurable disturbances.

The similarity between the simulated and experimental closed-loop responses is shown in Figure 6.26, which reproduces the experimental run of Figure 6.17 overplotted by the simulated response for a single +20% step change in feedflow. The one noticeable difference is the time delay in the experimental results that is not adequately described by the process model.

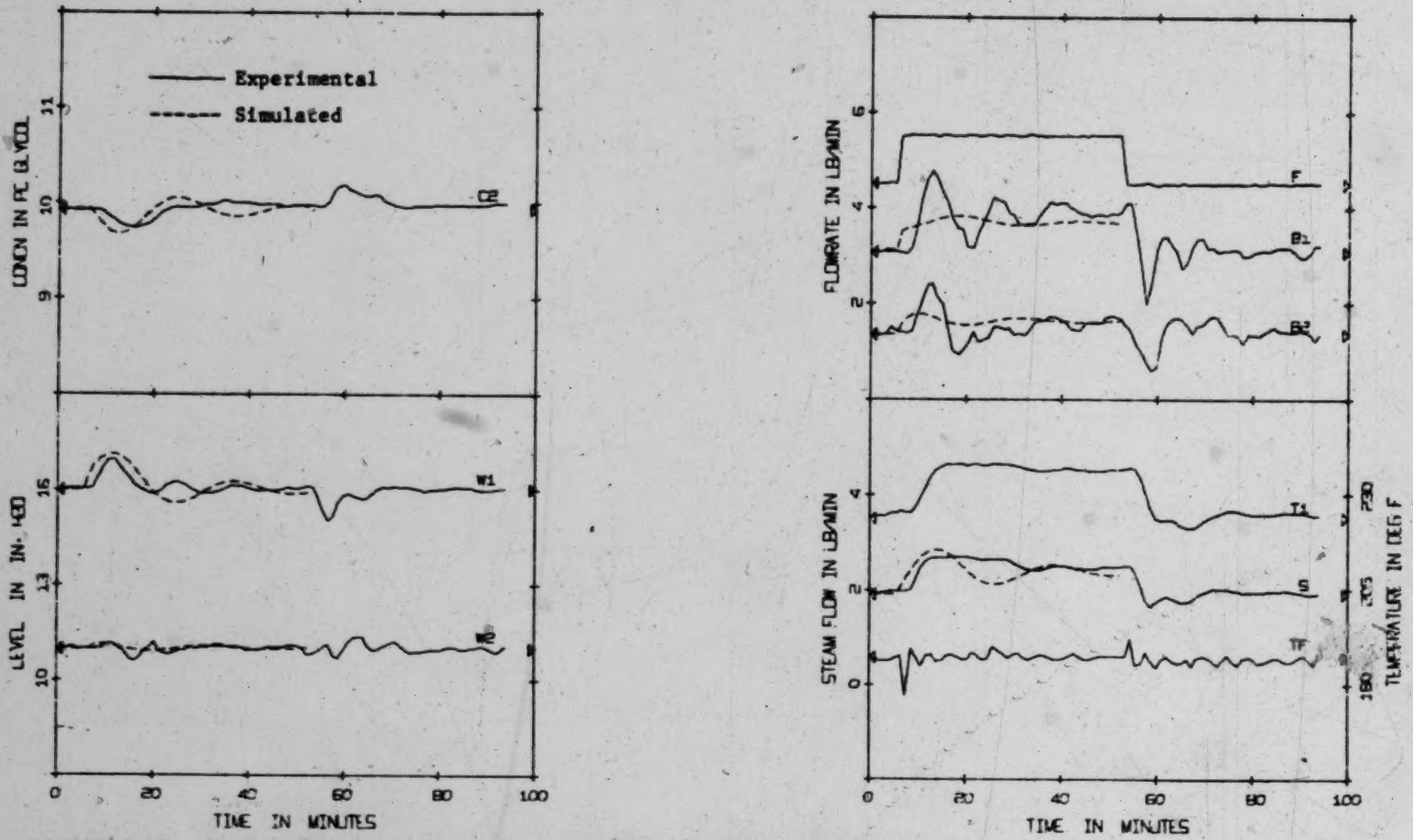


FIGURE 6.26 COMPARISON OF SIMULATED AND EXPERIMENTAL APPLICATION OF 3RED10(M) (PP, 10L(2018)/D, 20%F/FB + I/3RED10(M)/FB20, 3023B)

6.5 CONCLUSIONS

A unifying treatment for the methods of designing reduced order control laws to yield incomplete state feedback controllers is presented. A modal analysis, used to eliminate selected state variables from a high order control law produced a reduced order control law which controlled the pilot plant evaporator better than the other methods tried. This reduced order controller also compared favorably with the control using a fifth order control law.

The topic of model reduction is discussed. Most of the modal methods which have appeared in the literature can be shown to be one of two basic results or one of two revisions to one of these basic results.

CHAPTER SEVEN

RECOMMENDATIONS AND CONCLUSIONS

7.1 RECOMMENDED FUTURE WORK

As a result of this study of model reduction techniques and different approaches to the design of reduced order control laws, several areas for future work were identified.

7.1.1 GEMSCOPE

For the simulations in this thesis, GEMSCOPE was found to be a convenient package of programs. All the control law designs and simulated runs were calculated using GEMSCOPE. However, there are many design aids and techniques that have been developed that are not included as part of GEMSCOPE. Some of these are discussed in Chapter Two and include:

1. A general routine to calculate a linear state space model from a nonlinear model by a numerical linearization procedure
2. Provision for studying noisy processes
3. Provision for designing multivariable control laws by methods other than discrete dynamic programming

To make GEMSCOPE useful in a wider range of applications, it must be expanded to include these simulation and design techniques.

7.1.2 Least Squares Model Reduction Using Random Data

The approach to model reduction using random data with least squares resulted in acceptable reduced order continuous-time and discrete-time models of the evaporator. However, some difficulty was

experienced in the reduction of the twentieth order state space model of the department's distillation column. A summary of the work done in this area is described by the author [14]. A worthwhile future project would be to determine the source of the trouble described in this reference.

7.1.3 Least Squares Control Law Reduction Using Random Data

The use of least squares with random data to calculate a reduced order control law from a high order control law gave poor results. Suggested work in this area is to design the reduced order feedback control matrix using least squares plus random data, and then design the feedforward, integral and setpoint matrices to account for measurable disturbances, unmeasurable disturbances and setpoint changes. This approach for the design of feadforward control, given a feedback matrix, is considered by Newell et al [12].

7.1.4 "Implied Control"

An important area of control theory is to control a variable which cannot be measured on-line, and which requires a lengthy laboratory analysis. One approach to solving this problem could be the "implied control" technique outlined in Chapter Five. Perhaps, with a more detailed examination of the systems controllability, observability and structural characteristics a feasible "implied control" scheme designed by the method presented in Chapter Five, can be devised.

7.1.5 Controller Design Using a Model with Input Derivation Terms

One of the forms of the reduced order model calculated using a modal analysis contains an input derivative in its state equation as

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u} + \underline{\underline{E}}\,\underline{\dot{u}} \tag{7.1}$$

Marshall and Nicholson [10] present an interesting approach to obtain a simple control law from this model. However, since this control law is calculated from a continuous-time model, it must be applied as a continuous (analog) controller. Thus, it is not useful in controlling the process using a digital computer, unless the sampling interval is quite small.

By defining a new variable, $\bar{\underline{x}}$, Equation (7.1) can be rewritten as

$$\dot{\bar{\underline{x}}} = \underline{\underline{A}}\,\bar{\underline{x}} + \underline{\underline{\bar{B}}}\,\underline{u} \tag{7.2}$$

$$\underline{x} = \bar{\underline{x}} + \underline{\underline{E}}\,\underline{u} \tag{7.3}$$

and these can be written in the discrete-time form of

$$\bar{\underline{x}}(j+1) = \underline{\underline{\Phi}}\,\bar{\underline{x}}(j) + \underline{\underline{\bar{\Delta}}}\,\underline{u}(j) \tag{7.4}$$

$$\underline{x}(j) = \bar{\underline{x}}(j) + \underline{\underline{E}}\,\underline{u}(j) \tag{7.5}$$

An interesting project would be to extend the work of Marshall and Nicholson [10] to obtain a discrete-time control law from Equations (7.4) and (7.5), which would include the effect of $\underline{\underline{E}}\,\underline{u}(j)$ term in Equation (7.5).

7.1.6 Output Feedback

Most of the approaches to output feedback control [1 - 9, 13] are for continuous-time models, whose resulting control laws are also continuous-time. An extension of some of these approaches to design discrete-time controllers that could be applied to the control of a process using a digital computer would be useful. A direct comparison of the approaches used in this thesis with these other output feedback approaches could then be made.

7.1.7 Choice of $\underline{x}_1$ in Reduced Order Control Law

In the derivation of the reduced order control law it was assumed that $\underline{x}_1$ could satisfactorily stabilize the system. As was pointed out in Chapter Five, this is a reasonable assumption for the evaporator since $\underline{x}_1$ had been used in previous multi-loop control studies [11]. A better theoretical base for choosing $\underline{x}_1$ would be useful.

7.2 CONCLUSIONS

The GEneral Multipurpose Simulation and COntrol PackagE (GEMSCOPE) which was assembled for use in simulation and control system design studies, proved to be a convenient package of programs. All the simulations and control laws used in this thesis were calculated using GEMSCOPE.

The general problem of obtaining a reduced order discrete-time model from a high order continuous-time model was considered in detail. The required discrete-time model can be calculated by two alternate

approaches. These approaches were shown to be equivalent for several modal approaches to model reduction, but were not equivalent when a least squares model reduction approach was used.

The approach to model reduction using least squares with random data was more satisfactory than the more conventional trajectory fitting approach.

In general, selection of the "best" approach to model reduction depends upon the end use of the resulting reduced order model. Both the least squares with random data approach and the modal approach resulted in reduced order models which satisfactorily approximated the response of the high order model. Thus, they were satisfactory in open loop comparisons. Furthermore, proportional feedback, proportional feedback plus feedforward and proportional feedback plus setpoint control laws designed from these reduced order models gave adequate control of the pilot plant evaporator. However, not all of the reduced order models were good enough for use in the design of proportional plus integral feedback controllers.

The reduced order control law which controlled the pilot plant evaporator the best was that designed by the elimination of selected states from the high order control law. The controllers calculated by this approach were as good as, or better than the control laws calculated by other approaches when applied to the control of the pilot plant evaporator. Furthermore, this control law reduction approach was straightforward to use, requiring only a high order control law and a modal analysis of the system.

NOMENCLATURE

Alphabetic		Equation where first used
$\underline{A}$	Continuous-time state matrix (n x n)	(1.1)
$\underline{B}$	Continuous-time input (controls) matrix (n x m)	(1.1)
$\hat{\underline{B}}$	= ($\underline{B}$, $\underline{D}$) (n x (m + q))	(2.1)
$\underline{C}$	State to output matrix (p x n)	(1.2)
$\underline{D}$	Continuous-time input (disturbance) matrix (n x q)	(1.1)
$\underline{d}$	Disturbance vector (q x 1)	(1.1)
$\underline{E}_R$	Input to Output matrix (ℓ x m)	(3.34)
$\underline{E}^*$	Input plus delayed vector to output matrix	(2.2)
$\underline{F}$	State feedback matrix used in decoupling (m x n)	(2.6)
$\underline{G}$	Continuous-time canonical input (controls) matrix (n x m)	(5.22)
$\hat{\underline{G}}$	= ($\underline{G}$, $\underline{H}$) (n x (m + q))	(3.38)
$\underline{H}$	Continuous-time canonical input (disturbance) matrix (n x q)	(5.22)
$\underline{I}$	Identity matrix	(2.5)
$\underline{J}$	Continuous-time canonical state matrix (n x n)	(3.38)
J	Quadratic performance criterion	(2.11)
j	Time counter such that $\underline{x}(j)$ denotes $\underline{x}(t)$ at t = jT	(1.3)
$\underline{K}$	Control matrix	(2.8)
k	Number of data vectors in least squares solution	(4.8)
ℓ	Dimension of the reduced order state vector	
$\underline{M}$	Matrix to transform system into its canonical form (n x n)	(3.4)
m	Dimension of the control vector	

Nomenclature (Continued)

m	Integer	(2.4)
$\underline{\underline{N}}$	Matrix used in decoupling	(2.6)
N	Total number of time intervals in the quadratic performance criterion	(2.11)
n	Dimension of the state vector	
p	Dimension of the output vector	
$\underline{\underline{Q}}$	State weighting matrix used in the quadratic performance criterion	(2.11)
q	Dimension of the disturbance vector	
$\underline{\underline{R}}$	Control weighting matrix used in the quadratic performance criterion	(2.11)
$\underline{\underline{S}}$	Final state weighting matrix used in the quadratic performance criterion	(2.11)
$\underline{s}$	Integral contribution to the control law	(6.14)
s	Laplace transform variable	(2.7)
T	Discrete time interval	(2.4)
t	Time	(5.20)
$\underline{u}$	Control vector (m x 1)	(1.1)
$\hat{\underline{u}}$	$= \begin{bmatrix} \underline{u} \\ \underline{d} \end{bmatrix}$ ((m + q) x 1)	(2.1)
$\underline{\underline{V}}$	$= \underline{\underline{M}}^{-1}$	(3.8)
$\underline{v}$	Reference vector used in decoupling	(2.6)
$\underline{\underline{W}}$	Transfer matrix used in decoupling	(2.7)
$\underline{\underline{W}}$	Matrix used in the least squares solution	(4.6)
$\underline{w}$	Row of $\underline{\underline{W}}$ used in the least squares solution	(4.8)
$\underline{w}$	Input plus delayed vector	(2.2)
$\underline{x}$	State vector (n x 1)	(1.1)
$\underline{y}$	Output vector (p x 1)	(1.2)

Nomenclature (Continued)

$\underline{Z}$	Matrix used in the least squares solution	(4.6)
$\underline{z}$	Canonical state vector (n x 1)	(3.4)
$\underline{\mathbf{z}}$	Augmented vector used in least squares control law reduction	(6.46)

Greek

$\underline{\alpha}$	Discrete-time canonical state matrix (n x n)	(3.6)
β	Time weighting factor in the quadratic performance criterion	(2.11)
$\underline{\Delta}$	Discrete-time input (controls) matrix (n x m)	(1.3)
$\hat{\underline{\Delta}}$	$= (\underline{\Delta}, \underline{\Theta})$ (n x (m + q))	(2.3)
$\underline{\delta}$	Discrete-time canonical input (controls) matrix (n x m)	(5.9)
$\hat{\underline{\delta}}$	$= (\underline{\delta}, \underline{\eta})$ (n x (m + q))	(3.6)
ζ	Convergence criterion in least squares model reduction	(4.31)
$\underline{\eta}$	Discrete-time canonical input (disturbance) matrix (n x q)	(5.9)
$\underline{\Theta}$	Discrete-time input (disturbance) matrix (n x q)	(1.3)
$\underline{\xi}$	Vector used in Davison's Modal reduction	(3.20)
$\underline{\Phi}$	Discrete-time state matrix (n x n)	(1.3)

Subscripts

1,2,3,4	Refer to partitions of a matrix	(3.2)
I	Partition corresponding to the integrating states	(4.14)
i,j,k	Elements of a vector or matrix	
N	Partition corresponding to the non-integrating states	(4.14)
R	Refers to reduced order	(3.3)

Nomenclature (Continued)

Superscripts

FB	Proportional feedback	(2.8)
FF	Feedforward	(2.8)
I	Integral feedback	(2.8)
SP	Setpoint	(2.8)
T	Transpose of a vector or matrix	(2.11)
u	Intermediate matrix	(5.17)

Nomenclature for Computer Graphs

Each computer graph is identified by a string of characters. These are in the general form of

(Model or Pilot Plant/Run Conditions/Control Type/
Source of Control Law/Run Number)

The codes used are:

Model or Pilot Plant:

PP	Pilot Plant
nL(m)	Linear model (L) of order n calculated in run number m

Run Conditions:

D	Disturbance
SP	Set point
xx(0)	Initial condition of variable xx
+, -	Positive or negative change (no sign means both)
x %	Step size as a percentage of steady state
xx	Process variable disturbed

Nomenclature (Continued)

Type of Control:

OL	Open loop (no control)
FB	Proportional feedback control
FB + I	Proportional plus integral feedback control
FB + FF	Proportional feedback plus feedforward control
FB + SP	Proportional feedback plus setpoint control

Source of Control Law:

nOPT	Optimal control law for an n^{th} order model
3RED10	Third order control law reduced from the tenth order control law
DDC	Multiloop control

Run Number:

The simulated or experimental run in which the response was calculated.

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APPENDIX A

PILOT PLANT EVAPORATOR AND ITS MODELS

A.1 THE EVAPORATOR

The pilot plant evaporator used in this work is a double effect unit with the two effects operating in sequence. It was built by Andre [1]. The major pieces of process equipment are shown in the schematic diagram in Figure A.1. The control loops shown in Figure A.1 represent the multiloop control scheme applied to the evaporator in previous studies [1,6,10,11,13].

The first effect has natural circulation through its 18 inch long, 3/4 inch O.D. tubes. It is heated by process steam. The second effect is a long tube vertical unit which was run in its forced circulation mode. It has three, six foot long, one inch O.D. tubes. It is operated at a lower pressure than the first effect and is heated by the vapour produced in the first effect.

The evaporator is fully instrumented and can be controlled by either Foxboro electronic controllers or under Direct Digital Control (DDC) from an IBM 1800 Data Acquisition and Control Computer operating under MPX. Multiloop DDC can be applied directly using the computer system control package and advanced control schemes by user provided programs and a set of system programs to interface between the user and system control programs.

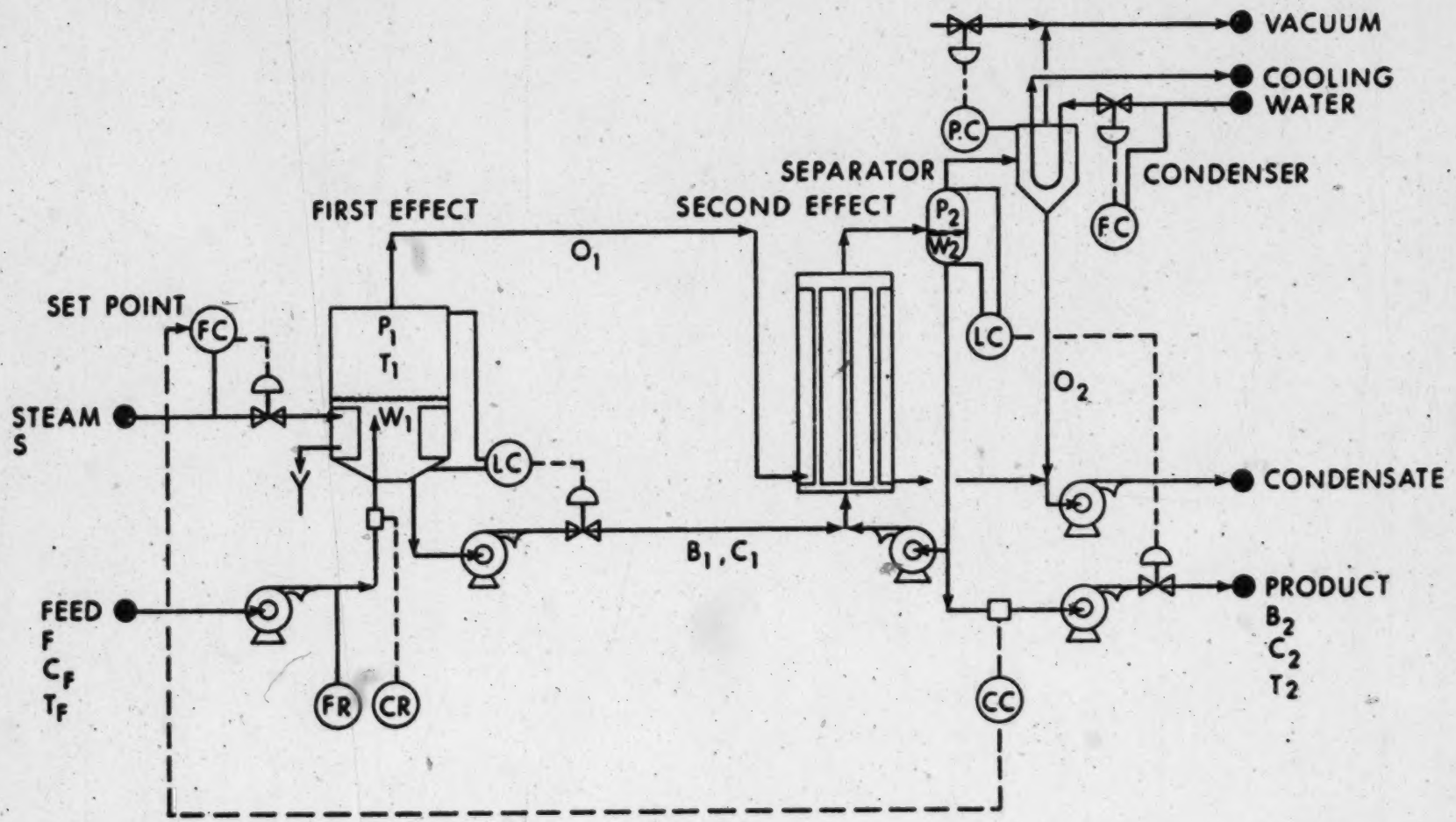


FIGURE A.1 PILOT PLANT DOUBLE EFFECT EVAPORATOR

A.2 THE EVAPORATOR MODEL

The evaporator model was first described by Andre [1]. Newell [11] presents a complete outline of a tenth order model in its nonlinear form. The model used in this work is a linearized form of Newell's nonlinear model, calculated using a numerical linearization procedure. The linearized model is in the continuous-time form of

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u} + \underline{\underline{D}}\,\underline{d} \tag{A.1}$$

$$\underline{y} = \underline{\underline{C}}\,\underline{x} \tag{A.2}$$

or in the discrete-time form of Equation (A.2) and

$$\underline{x}(j+1) = \underline{\underline{\Phi}}\,\underline{x}(j) + \underline{\underline{\Delta}}\,\underline{u}(j) + \underline{\underline{\Theta}}\,\underline{d}(j) \tag{A.3}$$

The elements of the vectors $\underline{x}$, $\underline{u}$, $\underline{d}$, $\underline{y}$ are defined as normalized perturbation variables as

$$x_3 = \frac{Wl - Wl_{ss}}{Wl_{ss}}$$

where Wl_{ss} is the normal steady state value of Wl. The vectors $\underline{x}$, $\underline{u}$, $\underline{d}$, and $\underline{y}$ are defined as follows

State Vector, $\underline{x}$:		Normal Steady State Value
TS	Steam Temperature (x_1)	244°F
TWl	First effect tube wall temperature (x_2)	227°F
Wl	First effect holdup (x_3)	45.5 lb.
Cl	First effect concentration (x_4)	4.59% glycol
Hl	First effect enthalpy (x_5)	189.2 BTU/lb

State Vector, $\underline{x}$: (continued)		Normal Steady State Value
TW2	Second effect tube wall temperature (x_6)	181°F
W2	Second effect holdup (x_7)	41.5 lb.
C2	Second effect concentration (x_8)	10.11% glycol
H2	Second effect enthalpy (x_9)	134.1 BTU/lb
TW3	Condenser tube wall temperature (x_{10})	108°F

Control Vector, $\underline{u}$:		
S	Steam flow (u_1)	2. lb/min
B1	First effect bottoms flow (u_2)	3.485 lb/min
B2	Second effect bottoms flow (u_3)	1.581 lb/min

Disturbance Vector, $\underline{d}$:		
F	Feed flow (d_1)	5. lb/min
CF	Feed concentration (d_2)	3.2% glycol
HF	Feed enthalpy (d_3)	156.9 BTU/lb

Output Vector, $\underline{y}$:

$\underline{y} = (W1, W2, C2)^T$

The coefficient matrices of the continuous-time model are shown in Table A.1 and those of the discrete-time model, with a 64 second time base, are shown in Table A.2. Table A.3 shows the system modal matrix, $\underline{M}$, along with the corresponding eigenvalues of both $\underline{A}$ and $\underline{\Phi}$.

The general open-loop model contains two integrating states (with eigenvalues of $\underline{\Phi}$ of unity). A closed loop model is also used

TABLE A.1

CONTINUOUS-TIME, OPEN-LOOP, TENTH ORDER EVAPORATOR MODEL (2016)

$\underline{A}$ =

-0.2978E 03	0.2724E 03	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
0.1043E 02	-0.4862E 02	0.0000E 00	0.2820E 00	0.3262E 02	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	-0.1348E 00	-0.5238E-06	-0.2818E-03	-0.3265E-01	0.1360E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.1348E 00	-0.1790E-04	-0.7630E-01	0.3265E-01	-0.1360E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.1126E 02	-0.3660E-06	-0.8823E-01	-0.1028E 02	0.6947E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00	0.8278E-01	0.9574E 01	-0.4552E 02	0.0000E 00	0.5587E 00	0.2738E 02	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00	-0.1148E-05	-0.1670E-03	-0.6412E-02	0.0000E 00	-0.2596E-02	-0.1273E 00	0.1021E 00
0.0000E 00	0.0000E 00	0.0000E 00	0.3811E-01	0.1670E-03	0.6412E-02	-0.2915E-04	-0.3548E-01	0.1273E 00	-0.1021E 00
0.0000E 00	0.0000E 00	0.0000E 00	-0.8138E-05	0.1171E 00	0.4505E 01	-0.1539E-05	-0.8959E-01	-0.4429E 01	0.7620E 00
0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.4205E-01	0.2062E 01	-0.8204E 01

$\underline{B}$ =

0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00
0.1100E 00	-0.7334E-05	-0.1028E-02
-0.3348E-01	0.7660E-01	0.1028E-02
-0.1765E-01	-0.3819E-04	0.8586E-01
0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00

$\underline{D}$ =

0.2053E 02	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	-0.7658E-01	0.0000E 00
0.0000E 00	-0.3482E-09	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.8291E-01	-0.3808E-01
0.0000E 00	-0.4579E-01	-0.8545E-08
0.0000E 00	0.3414E-01	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00

TABLE A.2

DISCRETE-TIME, OPEN-LOOP, TENTH ORDER EVAPORATOR MODEL (2018)

$\underline{\Phi}$ =

0.3836E-02	0.1092E 00	-0.2568E-06	0.3219E-02	0.3757E 00	0.1501E-01	-0.1414E-06	0.1911E-02	0.9376E-01	0.8400E-02
0.4188E-02	0.1193E 00	-0.2814E-06	0.3515E-02	0.4102E 00	0.1641E-01	-0.1553E-06	0.2090E-02	0.1024E 00	0.9192E-02
-0.6923E-03	-0.1978E-01	0.9998E 00	-0.5016E-03	-0.5753E-01	0.5370E-02	-0.4962E-07	0.6722E-03	0.3296E-01	0.2933E-02
0.6573E-03	0.1878E-01	-0.1835E-04	0.9217E 00	0.5487E-01	-0.5013E-02	0.4812E-07	-0.6369E-03	-0.3123E-01	-0.2799E-02
0.4961E-02	0.1413E 00	-0.1851E-06	-0.3800E-02	0.4860E 00	0.1967E-01	-0.1934E-06	0.2518E-02	0.1234E 00	0.1110E-01
0.2950E-02	0.8418E-01	-0.1406E-06	0.2438E-02	0.2919E 00	0.2297E-01	-0.7174E-06	0.3745E-02	0.1837E 00	0.1900E-01
-0.2556E-03	-0.7329E-02	0.5755E-08	-0.2180E-03	-0.2633E-01	-0.6145E-02	0.9998E 00	-0.1168E-02	-0.5731E-01	0.7482E-02
0.2687E-03	0.7704E-02	-0.3787E-06	0.3846E-01	0.2721E-01	0.5823E-02	-0.3054E-04	0.9610E 00	0.5503E-01	-0.7216E-02
0.3158E-02	0.9017E-01	-0.1113E-06	0.1806E-02	0.3140E 00	0.3064E-01	-0.5159E-06	-0.1436E-01	0.2560E 00	0.2725E-01
0.7696E-03	0.2197E-01	-0.2511E-07	0.6338E-03	0.7682E-01	0.8611E-02	-0.2671E-06	0.1498E-02	0.7357E-01	0.7963E-02

$\underline{\Delta}$ =

0.1857E 00	0.2200E-02	0.2128E-08
0.1280E 00	0.2417E-02	0.2346E-08
-0.9644E-02	-0.8090E-01	0.7565E-09
0.9347E-02	-0.7480E-03	-0.7402E-09
0.1338E 00	0.3009E-02	0.3003E-08
0.3457E-01	0.1116E-01	0.1088E-07
-0.2161E-02	0.8823E-01	-0.4063E-01
0.2283E-02	-0.4662E-01	0.6155E-06
0.4601E-01	0.1867E-01	0.1546E-07
0.9651E-02	0.4158E-02	0.5803E-08

$\underline{\Theta}$ =

-0.8858E-02	0.3069E-03	0.4238E-01
-0.9710E-02	0.3364E-03	0.4645E-01
0.1180E 00	-0.3028E-04	-0.4170E-02
-0.3491E-01	0.7847E-01	0.4038E-02
-0.1155E-01	-0.2673E-03	0.5671E-01
-0.4878E-02	0.1656E-03	0.2334E-01
0.2005E-03	-0.6683E-05	-0.9603E-03
-0.9045E-03	0.1601E-02	0.1013E-02
-0.4134E-02	0.1053E-03	0.1985E-01
-0.8757E-03	0.2913E-04	0.4192E-02

TABLE A.3

TENTH ORDER EVAPORATOR MODAL MATRIX AND EIGENVALUES OF $\underline{A}$ AND $\underline{\phi}$

$\underline{M}$ =

-0.3410E-07	-0.1097E-06	0.2825E-05	-0.1559E-03	-0.3852E 00	-0.1901E 00	-0.2514E-01	-0.8693E 00	0.8483E 00	0.9985E 00
-0.3253E-07	-0.1154E-06	0.6100E-05	-0.1711E-03	-0.4207E 00	-0.2070E 00	0.3171E-01	0.4336E 00	-0.1778E 00	0.1907E-01
-0.7428E 00	-0.7034E 00	-0.1962E-02	-0.3791E-04	-0.7805E-01	-0.5549E-01	0.1348E-02	0.5399E-03	-0.1810E-02	0.2844E-03
0.1741E-03	0.1650E-03	-0.9696E-05	0.7121E 00	0.1056E 00	0.5815E-01	-0.1383E-02	0.5311E-03	0.1826E-02	-0.2860E-03
-0.1552E-05	-0.1571E-05	0.9881E-05	-0.6361E-02	-0.5009E 00	-0.2372E 00	-0.2371E-02	-0.1209E 00	0.3609E-01	-0.1512E-01
-0.6031E-06	-0.6979E-06	0.1005E-04	-0.2021E-03	-0.3732E 00	0.4339E 00	-0.6787E-01	0.2029E 00	0.4948E 00	-0.4950E-01
0.6696E 00	0.7108E 00	0.3921E-03	-0.4630E-03	-0.1628E 00	0.4483E-01	-0.1363E-01	-0.2309E-03	0.7180E-04	-0.1180E-04
-0.3404E-03	-0.3814E-03	-0.9998E 00	-0.7019E 00	0.1717E 00	-0.4719E-01	0.1372E-01	0.1496E-03	-0.4512E-04	0.7214E-05
0.5949E-05	0.6661E-05	0.2041E-01	0.1406E-01	-0.4450E 00	0.7779E 00	-0.1046E 00	-0.1905E-01	-0.5072E-01	0.5352E-02
-0.2670E-06	-0.2985E-06	0.3026E-05	-0.6663E-04	-0.1150E 00	0.2465E 00	0.9912E 00	0.1018E-01	-0.5961E-02	0.8351E-03

Eigenvalues of $\underline{A}$

0.1905E-05	-0.1566E-04	-0.3807E-01	-0.7659E-01	-0.2876	-0.1705E 01	-0.8420E 01	-0.4656E 02	-0.4911E 02	-0.3089L 03

Eigenvalues of $\underline{\phi}$

0.9999E 00	0.9999E 00	0.9600E 00	0.9212E 00	0.7354E 00	0.1622E 00	0.1271E-03	0.2166E-04	0.6938E-05	-0.3728E-06

which changes the unity eigenvalues of $\underline{\Phi}$ to lie between zero and one. This is calculated, with a discrete-time form of the evaporator model which has an eight second time base, and the relations

$$\begin{aligned} B1 &= 0.9203 \quad W1 \\ B2 &= 2.768 \quad W2 \end{aligned} \tag{A.4}$$

to get the model shown in Table A.4. This model has an eight second time base.

The tenth order optimal control matrices for the model in Tables A.1 and A.2 have been calculated using dynamic programming as is outlined by Newell [11] and using the following weighting matrices. For proportional feedback, feedforward and setpoint control

$$\underline{Q} = \text{diag } [0, 0, 10, 0, 0, 0, 10, 100, 0, 0] \tag{A.5}$$

$$\underline{R} = \text{diag } [0.05, 0.05, 0.05] \tag{A.6}$$

and for proportional-plus integral feedback control, $\underline{R}$ is the same as Equation (A.6) while $\underline{Q}$ is given by

$$\underline{Q} = \text{diag } [0, 0, 10, 0, 0, 0, 10, 100, 0, 0, 0.5, 0.5, 1.0] \tag{A.7}$$

The controller matrices for this tenth order model are given in Table A.5.

A fifth order model is also used. It is in the general form of Equations (A.1) and (A.2) with a state vector of

$$\underline{x} = [W1, C1, H1, W2, C2]^T$$

TABLE A.4

DISCRETE-TIME, PARTIAL CLOSED-LOOP, TENTH ORDER EVAPORATOR MODEL (2107)

$\underline{\Phi}$ =

0.5934E-02	0.1687E 00	0.2937E-04	0.4927E-02	0.5702E 00	0.1012E-01	-0.2047E-08	0.3892E-03	0.1908E-01	0.5617E-03
0.6464E-02	0.1838E 00	0.3425E-04	0.5378E-02	0.6223E 00	0.1114E-01	-0.2387E-08	0.4400E-03	0.2157E-01	0.6503E-03
-0.1997E-03	-0.5783E-02	0.9906E 00	-0.8935E-04	-0.1032E-01	0.3483E-02	-0.9969E-09	0.1584E-03	0.7769E-02	0.2629E-03
0.1984E-03	0.5745E-02	-0.1665E-04	0.9898E 00	0.1027E-01	-0.3456E-02	0.9943E-09	-0.1577E-03	-0.7734E-02	-0.2620E-03
0.7537E-02	0.2147E 00	0.6200E-04	-0.2194E-02	0.7361E 00	0.1405E-01	-0.4315E-08	0.6646E-03	0.3258E-01	0.1130E-02
0.1890E-02	0.5436E-01	0.1793E-02	0.1707E-02	0.1991E 00	0.5365E-01	-0.1244E-06	0.9822E-02	0.4816E 00	0.2844E-01
-0.5277E-05	-0.1593E-03	0.1027E-01	-0.6639E-05	-0.7999E-03	-0.1312E-02	0.9859E 00	-0.2804E-03	-0.1374E-01	0.7759E-02
0.5900E-05	0.1778E-03	-0.5575E-02	0.5050E-02	0.8231E-03	0.1295E-02	-0.3855E-05	0.9951E 00	0.1369E-01	-0.7735E-02
0.6596E-03	0.1944E-01	0.3610E-02	0.5995E-03	0.8446E-01	0.7933E-01	-0.1638E-06	-0.4657E-02	0.7666E 00	0.5187E-01
0.5361E-04	0.1617E-02	0.3560E-03	0.6696E-04	0.8074E-02	0.1270E-01	-0.2474E-07	0.2862E-02	0.1403E 00	0.3409E 00

$\underline{\Delta}$ =

0.9546E-01
0.2944E-01
-0.3416E-03
0.3403E-03
0.1741E-01
0.3231E-02
-0.4178E-05
0.4826E-05
0.6984E-03
0.4141E-04

$\underline{\Theta}$ =

-0.1151E-02	0.4004E-04	0.5511E-02
-0.1296E-02	0.4508E-04	0.6205E-02
0.1468E-01	-0.1381E-05	-0.1927E-03
-0.4453E-02	0.1015E-01	0.1918E-03
-0.1867E-02	-0.2156E-04	0.9121E-02
-0.3804E-03	0.1316E-04	0.1820E-02
0.6545E-06	-0.2173E-07	-0.3133E-05
-0.1195E-04	0.2584E-04	0.3565E-05
-0.9607E-04	0.2675E-05	0.4610E-03
-0.6558E-05	0.2174E-06	0.3139E-04

TABLE A.5

OPTIMAL CONTROL LAWS FOR THE TENTH ORDER MODEL

Proportional plus, feedforward plus setpoint control

$\underline{K}^{FB} =$

$$\begin{bmatrix} -0.2438E-01 & -0.6964E\ 00 & 0.8790E\ 01 & -0.1711E\ 01 & -0.2304E\ 01 & -0.1072E\ 00 & -0.4185E-01 & -0.1530E\ 02 & -0.1027E\ 01 & 0.9727E-01 \\ 0.2031E-02 & 0.5833E-01 & 0.2547E\ 01 & 0.5391E\ 00 & 0.2371E\ 00 & 0.9395E-01 & -0.1307E\ 01 & 0.1163E\ 02 & 0.8415E\ 00 & -0.8783E-01 \\ 0.2804E-03 & 0.8053E-02 & 0.2722E\ 01 & 0.1117E\ 01 & 0.4854E-01 & 0.3186E-01 & 0.9751E\ 01 & 0.1215E\ 02 & 0.2624E\ 00 & -0.6441E-02 \end{bmatrix}$$

$$\underline{K}^{FF} = \begin{bmatrix} 0.3370E\ 01 & -0.2147E\ 00 & -0.4188E\ 00 \\ 0.4360E\ 00 & 0.3824E-01 & -0.9721E-03 \\ 0.8283E\ 00 & 0.1907E\ 00 & -0.1591E-02 \end{bmatrix}$$

$$\underline{K}^{SP} = \begin{bmatrix} -0.8791E\ 01 & 0.4340E-01 & 0.1721E\ 02 \\ -0.2547E\ 01 & 0.1307E\ 01 & -0.1219E\ 02 \\ -0.2722E\ 01 & -0.9755E\ 01 & -0.1346E\ 02 \end{bmatrix}$$

Proportional plus integral control

$\underline{K}^{FB} =$

$$\begin{bmatrix} -0.2973E-01 & -0.8492E\ 00 & 0.1213E\ 02 & -0.1983E\ 01 & -0.2798E\ 01 & -0.1281E\ 00 & 0.2329E\ 00 & -0.2024E\ 02 & -0.1268E\ 01 & 0.1411E\ 00 \\ 0.1333E-02 & 0.3841E-01 & 0.3133E\ 01 & 0.5214E\ 00 & 0.1743E\ 00 & 0.9276E-01 & -0.1261E\ 01 & 0.1108E\ 02 & 0.8215E\ 00 & -0.8202E-01 \\ -0.3982E-03 & -0.1133E-01 & 0.3905E\ 01 & 0.1225E\ 01 & -0.6425E-02 & 0.3968E-01 & 0.1223E\ 02 & 0.1476E\ 02 & 0.3168E\ 00 & -0.2946E-02 \end{bmatrix}$$

$$\underline{K}^{I} = \begin{bmatrix} 0.1693E\ 01 & 0.5076E-01 & -0.1807E\ 01 \\ 0.5681E\ 00 & -0.2919E\ 00 & 0.1079E\ 01 \\ 0.6092E\ 00 & 0.1932E\ 01 & 0.1352E\ 01 \end{bmatrix}$$

and coefficient matrices as shown in Table A.6. This model is the basic fifth order model used by Newell [11]. However, the coefficient matrices are different since it was linearized about a different steady state.

TABLE A.6

CONTINUOUS-TIME, OPEN-LOOP, FIFTH ORDER MODEL (2006)

$$\underline{A} = \begin{bmatrix} 0.0000E\ 00 & -0.1085E-02 & -0.1254E\ 00 & 0.0000E\ 00 & 0.0000E\ 00 \\ 0.0000E\ 00 & -0.7550E-01 & 0.1254E\ 00 & 0.0000E\ 00 & 0.0000E\ 00 \\ 0.0000E\ 00 & -0.6036E-02 & -0.7740E\ 00 & 0.0000E\ 00 & 0.0000E\ 00 \\ 0.0000E\ 00 & -0.1219E-02 & -0.1447E\ 00 & 0.0000E\ 00 & 0.1258E-03 \\ 0.0000E\ 00 & 0.3930E-01 & 0.1447E\ 00 & 0.0000E\ 00 & -0.3796E-01 \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} 0.0000E\ 00 & -0.7658E-01 & 0.0000E\ 00 \\ 0.0000E\ 00 & 0.0000E\ 00 & 0.0000E\ 00 \\ 0.2159E\ 00 & 0.0000E\ 00 & 0.0000E\ 00 \\ 0.0000E\ 00 & 0.7945E-01 & -0.3808E-01 \\ 0.0000E\ 00 & -0.4135E-01 & 0.0000E\ 00 \end{bmatrix}$$

$$\underline{D} = \begin{bmatrix} 0.1098E\ 00 & 0.0000E\ 00 & 0.0000E\ 00 \\ -0.3328E-01 & 0.7658E-01 & 0.0000E\ 00 \\ -0.1877E-01 & 0.0000E\ 00 & 0.9111E-01 \\ 0.0000E\ 00 & 0.0000E\ 00 & 0.0000E\ 00 \\ 0.0000E\ 00 & 0.0000E\ 00 & 0.0000E\ 00 \end{bmatrix}$$

APPENDIX B

CALCULATION OF A REDUCED ORDER DISCRETE-TIME MODEL FROM A HIGH ORDER CONTINUOUS-TIME MODEL BY TWO APPROACHES

A reduced order discrete-time model can be calculated from a high order continuous-time model by two approaches, as is discussed in Chapters Three and Four and shown in Figures 3.1 and 4.3. Using the modal analysis the same result is calculated while using the least squares analysis different results are obtained. These conclusions are shown in this Appendix. In this Appendix, the elements in the control and the disturbance vectors will both be included in one vector, called $\underline{u}$.

B.1 EQUIVALENCE OF RESULTS BY MARSHALL'S ANALYSIS

B.1.1 $\underline{\Phi}_R$ Derived from Path I ($\underline{\Phi}_R(I)$)

By definition, $\underline{\Phi}_R(I) = e^{\underline{A}_R T}$

By Equation (3.42) $\underline{A}_R = \underline{M}_1 \underline{J}_1 \underline{M}_1^{-1}$

Thus, using the theorem in Appendix D.1

$$\underline{\Phi}_R(I) = \underline{M}_1 e^{\underline{J}_1 T} \underline{M}_1^{-1} \qquad (B.1)$$

B.1.2 $\underline{\Phi}_R$ Derived from Path II ($\underline{\Phi}_R(II)$)

By path II, Equation (3.13)

$$\underline{\Phi}_R(II) = \underline{\Phi}_1 - \underline{\Phi}_2 \underline{V}_4^{-1} \underline{V}_3$$

Using $\underline{\Phi}_1$ and $\underline{\Phi}_2$ from Appendix D.3

$$\underline{\Phi}_R(II) = \underline{M}_1 \, e^{\underline{J}_1^T} \, (\underline{V}_1 - \underline{V}_2 \, \underline{V}_4^{-1} \, \underline{V}_3)$$

Using Equation (D.19)

$$\underline{\Phi}_R(II) = \underline{M}_1 \, e^{\underline{J}_1^T} \, \underline{M}_1^{-1} \qquad (B.2)$$

Thus, $\underline{\Phi}_R$ by Marshall's analysis is the same using the two approaches of Figure 3.1.

B.1.3 $\underline{\Delta}_R$ Derived from Path I ($\underline{\Delta}_R(I)$)

By definition, $\underline{\Delta}_R(I) = \int_0^T e^{\underline{A}_R(T-\tau)} \, \underline{B}_R \, d\tau$

Using Equations (3.42) and (3.43) and the theorem in Appendix D.1

$$\underline{\Delta}_R(I) = \int_0^T \left\{ \underline{M}_1 \, e^{\underline{J}_1(T-\tau)} \, \underline{M}_1^{-1} \right\} \left\{ \underline{B}_1 - \underline{A}_2 \, \underline{V}_4^{-1} \, \underline{J}_2^{-1} \, (\underline{V}_3 \, \underline{B}_1 + \underline{V}_4 \, \underline{B}_2) \right\} d\tau \qquad (B.3)$$

B.1.4 $\underline{\Delta}_R$ Derived from Path II ($\underline{\Delta}_R(II)$)

By Path II, Equation (3.14) gives

$$\underline{\Delta}_R(II) = \underline{\Delta}_1 + \underline{\Phi}_2 \, \underline{V}_4^{-1} \, (\underline{I} - \underline{\alpha}_2)^{-1} \, \underline{\delta}_2$$

Using the relations for $\underline{\Delta}_1$ and $\underline{\Phi}_2$ from Appendices D.6 and D.3, and $\underline{\alpha}_2$ and $\underline{\delta}_2$ from Appendix D.7,

$$\underline{\Delta}_R(II) = (\underline{M}_1, \underline{M}_2) \int_0^T e^{\underline{J}(T-\tau)} \underline{V}\, \underline{B}\, d\tau$$

$$+ (\underline{M}_1, \underline{M}_2)\, e^{\underline{J}T} \begin{pmatrix} \underline{V}_2 \\ \underline{V}_4 \end{pmatrix} \underline{V}_4^{-1} \left(\underline{I} - e^{\underline{J}_2 T} \right)^{-1} (\underline{V}_3, \underline{V}_4) \int_0^T \underline{M}\, e^{\underline{J}(T-\tau)}\, \underline{V}\, \underline{B}\, d\tau$$

$$- (\underline{M}_1, \underline{M}_2) \left\{ \int_0^T \begin{bmatrix} e^{\underline{J}_1(T-\tau)} & \underline{0} \\ \underline{0} & e^{\underline{J}_2(T-\tau)} \end{bmatrix} \begin{bmatrix} \underline{V}_1 & \underline{V}_2 \\ \underline{V}_3 & \underline{V}_4 \end{bmatrix} \begin{bmatrix} \underline{B}_1 \\ \underline{B}_2 \end{bmatrix} d\tau \right.$$

$$+ e^{\underline{J}T} \begin{pmatrix} \underline{V}_2 \\ \underline{V}_4 \end{pmatrix} \underline{V}_4^{-1} \left(\underline{I} - e^{\underline{J}_2 T} \right)^{-1} (\underline{V}_3\, \underline{M}_1 + \underline{V}_4\, \underline{M}_3,\ \underline{V}_3\, \underline{M}_2 + \underline{V}_4\, \underline{M}_4) \int_0^T e^{\underline{J}(T-\tau)}$$

$$\left. \underline{V}\, \underline{B}\, d\tau \right\}$$

This can be rearranged using relations in Appendix D.4 to get

$$\underline{\Delta}_R(II) = (\underline{M}_1, \underline{M}_2) \left\{ \int_0^T \begin{pmatrix} e^{\underline{J}_1(T-\tau)} \underline{V}_1\, \underline{B}_1 + e^{\underline{J}_1(T-\tau)} \underline{V}_2\, \underline{B}_2 \\ e^{\underline{J}_2(T-\tau)} \underline{V}_3\, \underline{B}_1 + e^{\underline{J}_2(T-\tau)} \underline{V}_4\, \underline{B}_2 \end{pmatrix} d\tau \right.$$

$$\left. + \begin{bmatrix} e^{\underline{J}_1 T} \underline{V}_2\, \underline{V}_4^{-1} \\ e^{\underline{J}_2 T} \end{bmatrix} \left(\underline{I} - e^{\underline{J}_2 T} \right)^{-1} \int_0^T e^{\underline{J}_2(T-\tau)} (\underline{V}_3\, \underline{B}_1 + \underline{V}_4\, \underline{B}_2)\, d\tau \right\}$$

$$= \underline{M}_1 \int_0^T e^{\underline{J}_1(T-\tau)} \underline{V}_1\, \underline{B}_1\, d\tau + \underline{M}_1 \int_0^T e^{\underline{J}_1(T-\tau)} \underline{V}_2\, \underline{B}_2\, d\tau + \underline{L}_1 + \underline{L}_2$$

where $\underline{L}_1 = \underline{M}_1\, e^{\underline{J}_1 T}\, \underline{V}_2\, \underline{V}_4^{-1} \left(\underline{I} - e^{\underline{J}_2 T} \right)^{-1} \int_0^T e^{\underline{J}_2(T-\tau)} (\underline{V}_3\, \underline{B}_1 + \underline{V}_4\, \underline{B}_2)\, d\tau$

$$\text{and } \underline{L}_2 = \underline{M}_2 \left(\underline{I} + e^{\underline{J}_2 T} \left(\underline{I} - e^{\underline{J}_2 T} \right)^{-1} \right) \int_0^T e^{\underline{J}_2 (T-\tau)} \left(\underline{V}_3 \underline{B}_1 + \underline{V}_4 \underline{B}_2 \right) d\tau$$

Expressions for $\underline{V}_1$ and $\underline{V}_2$ in Equations (D.16) and (D.17) give

$$\underline{\Delta}_R(II) = \underline{M}_1 \int_0^T e^{\underline{J}_1 (T-\tau)} \underline{M}_1^{-1} \left(\underline{I} - \underline{M}_2 \underline{V}_3 \right) \underline{B}_1 \, d\tau$$

$$+ \underline{M}_1 \int_0^T e^{\underline{J}_1 (T-\tau)} \left(-\underline{M}_1^{-1} \underline{M}_2 \underline{V}_4 \right) \underline{B}_2 \, d\tau + \underline{L}_1 + \underline{L}_2$$

$$= \underline{L}_3 - \underline{M}_1 \int_0^T e^{\underline{J}_1 (T-\tau)} \underline{M}_1^{-1} \underline{M}_2 \left(\underline{V}_3 \underline{B}_1 + \underline{V}_4 \underline{B}_2 \right) d\tau + \underline{L}_1 + \underline{L}_2$$

$$\text{where } \underline{L}_3 = \underline{M}_1 \int_0^T e^{\underline{J}_1 (T-\tau)} \underline{M}_1^{-1} \underline{B}_1 \, d\tau$$

Also, using the expression for $\underline{M}_2$ in Equation (D.18) gives

$$\underline{\Delta}_R(II) = \underline{L}_3 + \underline{L}_4 + \underline{L}_5 + \underline{L}_1 + \underline{L}_2$$

$$\text{where } \underline{L}_4 = -\int_0^T \underline{M}_1 e^{\underline{J}_1 (T-\tau)} \underline{M}_1^{-1} \underline{A}_2 \underline{V}_4^{-1} \underline{J}_2^{-1} \left(\underline{V}_3 \underline{B}_1 + \underline{V}_4 \underline{B}_2 \right) d\tau$$

$$\text{and } \underline{L}_5 = \int_0^T \underline{M}_1 e^{\underline{J}_1 (T-\tau)} \underline{J}_1 \underline{V}_2 \underline{V}_4^{-1} \underline{J}_2^{-1} \left(\underline{V}_3 \underline{B}_1 + \underline{V}_4 \underline{B}_2 \right) d\tau$$

Now consider

$$\underline{L}_3 + \underline{L}_4 = \int_0^T \left\{ \underline{M}_1 e^{\underline{J}_1 (T-\tau)} \underline{M}_1^{-1} \right\} \left\{ \underline{B}_1 - \underline{A}_2 \underline{V}_4^{-1} \underline{J}_2^{-1} \left(\underline{V}_3 \underline{B}_1 + \underline{V}_4 \underline{B}_2 \right) \right\} d\tau$$

$$= \underline{\Delta}_R(I) \quad \text{by Equation (B.3)}$$

Thus $\underline{\Delta}_R(II) = \underline{\Delta}_R(I) + \underline{L}_5 + \underline{L}_1 + \underline{L}_2$

Performing the integration on terms $\underline{L}_1$ and $\underline{L}_2$ using the result of Appendix D.9 and on term $\underline{L}_5$ using the result of Appendix D.10 gives

$$\underline{L}_1 = -\underline{M}_1 e^{\underline{J}_1^T} \underline{V}_2 \underline{V}_4^{-1} \underline{J}_2^{-1} (\underline{V}_3 \underline{B}_1 + \underline{V}_4 \underline{B}_2)$$

$$\underline{L}_2 = -\underline{M}_2 \left(\underline{I} + e^{\underline{J}_2^T} \left(\underline{I} - e^{\underline{J}_2^T}\right)^{-1}\right)\left(\underline{I} - e^{\underline{J}_2^T}\right) \underline{J}_2^{-1} (\underline{V}_3 \underline{B}_1 + \underline{V}_4 \underline{B}_2)$$

$$\underline{L}_5 = -\underline{M}_1 \left(\underline{I} - e^{\underline{J}_1^T}\right) \underline{V}_2 \underline{V}_4^{-1} \underline{J}_2^{-1} (\underline{V}_3 \underline{B}_1 + \underline{V}_4 \underline{B}_2)$$

Therefore,

$$\underline{\Delta}_R(II) = \underline{\Delta}_R(I) + \left\{ -\underline{M}_1 \underline{V}_2 \underline{V}_4^{-1} + \underline{M}_1 e^{\underline{J}_1^T} \underline{V}_2 \underline{V}_4^{-1} - \underline{M}_1 e^{\underline{J}_1^T} \underline{V}_2 \underline{V}_4^{-1} - \underline{M}_2 \left(\underline{I} - e^{\underline{J}_2^T}\right) - \underline{M}_2 e^{\underline{J}_2^T} \right\} \underline{J}_2^{-1} (\underline{V}_3 \underline{B}_1 + \underline{V}_4 \underline{B}_2)$$

$$\underline{\Delta}_R(II) = \underline{\Delta}_R(I) - (\underline{M}_1 \underline{V}_2 \underline{V}_4^{-1} + \underline{M}_2) \underline{J}_2^{-1} (\underline{V}_3 \underline{B}_1 + \underline{V}_4 \underline{B}_2)$$

and using the relation for $\underline{V}_2$ in Equation (D.17)

$$\underline{\Delta}_R(II) = \underline{\Delta}_R(I)$$

Thus, the two approaches using Marshall's analysis give identical reduced order discrete-time models.

B.2 EQUIVALENCE OF RESULTS BY DAVISON'S ANALYSIS

B.2.1 $\underline{\Phi}_R$ Derived from Path I ($\underline{\Phi}_R(I)$)

Davison's result for $\underline{A}_R$ using Equation (3.44) is

$$\underline{A}_R = \underline{A}_1 + \underline{A}_2 \underline{M}_3 \underline{M}_1^{-1}$$

The expressions for $\underline{A}_1$ and $\underline{A}_2$ in Appendix D.2 give

$$\underline{A}_R = \underline{M}_1 \underline{J}_1 \left[\underline{V}_1 + \underline{V}_2 \underline{M}_3 \underline{M}_1^{-1}\right] + \underline{M}_2 \underline{J}_2 \left[\underline{V}_3 + \underline{V}_4 \underline{M}_3 \underline{M}_1^{-1}\right]$$

Equations (D.5) and (D.7) allow $\underline{A}_R$ to be written as

$$\underline{A}_R = \underline{M}_1 \underline{J}_1 \underline{M}_1^{-1} \qquad (B.4)$$

Then, by the definition of $\underline{\Phi}_R$ and the theorem of Appendix D.1,

$$\underline{\Phi}_R(I) = e^{\underline{A}_R^T} = \underline{M}_1 e^{\underline{J}_1^T} \underline{M}_1^{-1} \qquad (B.5)$$

B.2.2 $\underline{\Phi}_R$ Derived from Path II ($\underline{\Phi}_R(II)$)

$\underline{\Phi}_R$ from path II, (Equation (3.22)) is

$$\underline{\Phi}_R(II) = \underline{\Phi}_1 + \underline{\Phi}_2 \underline{M}_3 \underline{M}_1^{-1}$$

Using the expressions for $\underline{\Phi}_1$ and $\underline{\Phi}_2$ in Appendix D.3, this can be written as

$$\underline{\Phi}_R(II) = \underline{M}_1 e^{\underline{J}_1^T} (\underline{V}_1 + \underline{V}_2 \underline{M}_3 \underline{M}_1^{-1}) + \underline{M}_2 e^{\underline{J}_2^T} (\underline{V}_3 + \underline{V}_4 \underline{M}_3 \underline{M}_1^{-1})$$

Making use of Equations D.5 and D.7, this is written as

$$\underline{\Phi}_R(II) = \underline{M}_1 e^{\underline{J}_1^T} \underline{M}_1^{-1}$$

which is identical to Equation (B.5)

B.2.3 $\underline{\Delta}_R$ Derived from Path (I) ($\underline{\Delta}_R(I)$)

$\underline{\Delta}_R(I)$ by definition is

$$\underline{\Delta}_R(I) = \int_0^T e^{\underline{A}_R(T-\tau)} \underline{B}_R \, d\tau$$

Using Equations (B.4), (3.45) and the theorem of Appendix D.1

$$\underline{\Delta}_R(I) = \underline{M}_1 \int_0^T e^{\underline{J}_1(T-\tau)} (\underline{V}_1 \underline{B}_1 + \underline{V}_2 \underline{B}_2) \, d\tau \qquad (B.6)$$

B.2.4 $\underline{\Delta}_R$ Derived from Path (II) ($\underline{\Delta}_R(II)$)

Equation (3.25) gives

$$\underline{\Delta}_R(II) = \underline{M}_1 (\underline{V}_1 \underline{\Delta}_1 + \underline{V}_2 \underline{\Delta}_2)$$

Using the definition of $\underline{\Delta}$ in Appendix (D.6)

$$\underline{\Delta}_R(II) = \underline{M}_1 (\underline{V}_1, \underline{V}_2) \int_0^T \underline{M} e^{\underline{J}(T-\tau)} \underline{M}^{-1} \underline{B} \, d\tau$$

$$= \underline{M}_1 (\underline{V}_1 \underline{M}_1 + \underline{V}_2 \underline{M}_3, \underline{V}_1 \underline{M}_2 + \underline{V}_2 \underline{M}_4) \int_0^T e^{\underline{J}(T-\tau)} \underline{M}^{-1} \underline{B} \, d\tau$$

Equations (D.5) and (D.6) give

$$\underline{\Delta}_R(II) = \underline{M}_1 (\underline{I}, \underline{0}) \int_0^T e^{\underline{J}(T-\tau)} \underline{M}^{-1} \underline{B} \, d\tau$$

so that $$\underline{\Delta}_R(II) = \underline{M}_1 \int e^{\underline{J}_1(T-\tau)} (\underline{V}_1 \underline{B}_1 + \underline{V}_2 \underline{B}_2) \, d\tau$$

which is identical to Equation (B.6).

Thus, Davison's analysis produces identical reduced order discrete-time models by the two approaches.

B.3 EQUIVALENCE OF RESULTS BY FOSSARD'S ANALYSIS

Fossard's analysis produces the same $\underline{\underline{A}}_R$, $\underline{\underline{B}}_R$, $\underline{\underline{\Phi}}_R$ and $\underline{\underline{\Delta}}_R$ as does Davison's analysis. The equivalence of this $\underline{\underline{\Phi}}_R$ and $\underline{\underline{\Delta}}_R$ by the two approaches in Figure 3.1 was shown in Appendix B.2. The equivalence of the $\underline{\underline{E}}_R$ matrix by the two approaches must now be shown.

B.3.1 $\underline{\underline{E}}_R$ Derived from Path I ($\underline{\underline{E}}_R(I)$)

From Equation (3.48)

$$\underline{\underline{E}}_R(I) = -\underline{\underline{M}}_2 \, \underline{\underline{J}}_2^{-1} \, \underline{\underline{G}}_2 \tag{B.7}$$

B.3.2 $\underline{\underline{E}}_R$ Derived from Path II ($\underline{\underline{E}}_R(II)$)

From Equation (3.35), using Equations (D.14) and (D.15) and Appendix D.6,

$$\underline{\underline{E}}_R(II) = \underline{\underline{M}}_2 \, (\underline{\underline{I}} - \underline{\underline{\alpha}}_2)^{-1} \, \underline{\underline{\phi}}_2$$

$$= \underline{\underline{M}}_2 \left(\underline{\underline{I}} - e^{\underline{\underline{J}}_2 T} \right)^{-1} (\underline{\underline{V}}_3, \underline{\underline{V}}_4) \, \underline{\underline{\Delta}}$$

$$= \underline{\underline{M}}_2 \left(\underline{\underline{I}} - e^{\underline{\underline{J}}_2 T} \right)^{-1} (\underline{\underline{V}}_3, \underline{\underline{V}}_4) \, \underline{\underline{M}} \int_0^T e^{\underline{\underline{J}}(T-\tau)} \, \underline{\underline{V}} \, \underline{\underline{B}} \, d\tau$$

$$= \underline{\underline{M}}_2 \left(\underline{\underline{I}} - e^{\underline{\underline{J}}_2 T} \right)^{-1} (\underline{\underline{V}}_3 \, \underline{\underline{M}}_1 + \underline{\underline{V}}_4 \, \underline{\underline{M}}_3, \underline{\underline{V}}_3 \, \underline{\underline{M}}_2 + \underline{\underline{V}}_4 \, \underline{\underline{M}}_4) \int_0^T e^{\underline{\underline{J}}(T-\tau)} \, \underline{\underline{G}} \, d\tau$$

Making use of Equations (D.7) and (D.8)

$$\underline{E}_R(II) = \underline{M}_2 \left(\underline{I} - e^{\underline{J}_2 T} \right)^{-1} (\underline{0}, \underline{I}) \int_0^T e^{\underline{J}(T-\tau)} \underline{G}\, d\tau$$

$$= \underline{M}_2 \left(\underline{I} - e^{\underline{J}_2 T} \right)^{-1} \int_0^T e^{\underline{J}_2 (T-\tau)} \underline{G}_2\, d\tau$$

The integral in Appendix D.9 is used to obtain

$$\underline{E}_R(II) = \underline{M}_2 \left(\underline{I} - e^{\underline{J}_2 T} \right)^{-1} \left(e^{\underline{J}_2 T} - \underline{I} \right) \underline{J}_2^{-1} \underline{G}_2$$

Thus,

$$\underline{E}_R(II) = -\underline{M}_2 \underline{J}_2^{-1} \underline{G}_2$$

which is identical to Equation (B.7) above. Thus, Fossard's analysis produces identical reduced order discrete-time models by the two approaches of Figure 3.1.

B.4 DIFFERENCE OF RESULTS BY LEAST SQUARES

If the same data, (trajectory or random numbers) are used for both the continuous-time reduction and the discrete-time reduction of the two paths in Figure 4.3, the resulting reduced order models will, in general, be different. This can be shown as follows

Equation (4.25) shows that $\underline{A}_R^T = (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T \underline{W}_c$

where $\underline{W}_c$ is calculated from the continuous-time model. Each row of $\underline{W}_c$ is

$$\dot{\underline{x}}_1^T = \underline{x}_1^T \underline{A}_1^T + \underline{x}_2^T \underline{A}_2^T$$

so that

$$\underline{W}_c = \underline{Z}\, \underline{A}_1^T + \underline{X}_2\, \underline{A}_2^T$$

where $\underline{X}_2$ is the matrix made up of the data in the $\underline{x}_2(j)$ vectors as

$$\underline{X}_2 = \begin{bmatrix} \underline{x}_2^T(1) \\ \underline{x}_2^T(2) \\ \vdots \\ \underline{x}_2^T(k) \end{bmatrix}$$

Thus,

$$\underline{A}_R^T = (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T (\underline{Z} \underline{A}_1^T + \underline{X}_2 \underline{A}_2^T)$$

$$= (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T (\underline{Z}, \underline{X}_2) \begin{pmatrix} \underline{A}_1^T \\ \underline{A}_2^T \end{pmatrix} = \underline{K} \begin{pmatrix} \underline{A}_1^T \\ \underline{A}_2^T \end{pmatrix}$$

where

$$\underline{K} = (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T (\underline{Z}, \underline{X}_2)$$

Thus,

$$\underline{A}_R = (\underline{A}_1, \underline{A}_2) \underline{K}^T$$

Similarly, for the discrete-time reduction, Equation (4.7) is

$$\underline{\Phi}_R = (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T \underline{W}_D$$

where Z contains the same data as for the continuous case above and $\underline{W}_D$ is calculated from the discrete-time model. Each row of $\underline{W}_D$ is

$$\underline{x}_1^T(j + 1) = \underline{x}_1^T(j) \underline{\Phi}_1^T + \underline{x}_2^T(j) \underline{\Phi}_2^T$$

so that

$$\underline{W}_D = \underline{Z} \underline{\Phi}_1^T + \underline{X}_2 \underline{\Phi}_2^T$$

Now we can arrange $\underline{\Phi}_R$ like we did $\underline{A}_R$ to give

$$\underline{\Phi}_R = (\underline{\Phi}_1, \underline{\Phi}_2)\ \underline{K}^T$$

This can be expanded using the following relation of $\underline{\Phi}$ and $\underline{\alpha}$ and relations in Appendix D.3 ($\underline{\alpha} = \exp(\underline{J}T)$)

$$\underline{\Phi} = \underline{M}\ \underline{\alpha}\ \underline{V}$$

$$(\underline{\Phi}_1, \underline{\Phi}_2) = (\underline{M}_1, \underline{M}_2)\ \underline{\alpha}\ \underline{V} = (\underline{M}_1, \underline{M}_2)\ \exp(\underline{J}T)\ \underline{V}$$

Thus,

$$\underline{\Phi}_R = (\underline{M}_1, \underline{M}_2)\ \exp(\underline{J}T)\ \underline{V}\ \underline{K}^T \qquad (B.8)$$

The expression for $\underline{A}_R$ can similarly be expanded into

$$\underline{A}_R = (\underline{M}_1, \underline{M}_2)\ \underline{J}\ \underline{V}\ \underline{K}^T$$

The discrete-time model corresponding to this $\underline{A}_R$ is

$$\underline{\Phi}_R = \exp(\underline{A}_R T) = \exp\left\{(\underline{M}_1, \underline{M}_2)\ \underline{J}T\underline{V}\ \underline{K}^T\right\}$$

This expression only equals the Expression (B.8) if $\underline{V}\ \underline{K}^T\ (\underline{M}_1, \underline{M}_2) = \underline{I}$, as is shown in Appendix D.12.

Expand this expression using

$$\underline{K} = (\underline{Z}^T\ \underline{Z})^{-1}\ \underline{Z}^T\ (\underline{Z}, \underline{X}_2) = (\underline{I}, (\underline{Z}^T\ \underline{Z})^{-1}\ \underline{Z}^T\ \underline{X}_2) = (\underline{I}, \underline{K}_2)$$

Thus,

$$\underline{V}\ \underline{K}^T\ (\underline{M}_1, \underline{M}_2) = \underline{V}\begin{pmatrix}\underline{I} \\ \underline{K}_2^T\end{pmatrix}(\underline{M}_1, \underline{M}_2) = \begin{pmatrix}\underline{V}_1 & \underline{V}_2 \\ \underline{V}_3 & \underline{V}_4\end{pmatrix}\begin{pmatrix}\underline{M}_1 & \underline{M}_2 \\ \underline{K}_2^T\ \underline{M}_1 & \underline{K}_2^T\ \underline{M}_2\end{pmatrix}$$

This is only equal to the identity matrix if

$$\underline{\underline{K}}_2^T \underline{\underline{M}}_1 = \underline{\underline{M}}_3 \quad \text{and} \quad \underline{\underline{K}}_2^T \underline{\underline{M}}_2 = \underline{\underline{M}}_4 \tag{B.9}$$

$\underline{\underline{K}}_2^T$ can be eliminated from Equations (B.9) to give

$$\underline{\underline{M}}_3 \underline{\underline{M}}_1^{-1} \underline{\underline{M}}_2 = \underline{\underline{M}}_4$$

which can be rearranged, using Equation (D.7) to give

$$\underline{\underline{V}}_3 \underline{\underline{M}}_2 + \underline{\underline{V}}_4 \underline{\underline{M}}_4 = \underline{\underline{0}} \tag{B.10}$$

However, Equation (D.8) says that

$$\underline{\underline{V}}_3 \underline{\underline{M}}_2 + \underline{\underline{V}}_4 \underline{\underline{M}}_4 = \underline{\underline{I}}$$

so that Equation (B.10) and thus, Equation (B.9) cannot hold. Thus, $\underline{\underline{V}}\underline{\underline{K}}^T (\underline{\underline{M}}_1, \underline{\underline{M}}_2) \neq \underline{\underline{I}}$ and different discrete-time models are calculated by the two paths shown in Figure 4.3.

APPENDIX C

MODEL REDUCTION - EQUIVALENCE OF FORMS

Many of the modal reduction results presented in the literature are equivalent. This Appendix will show these equivalences. In this Appendix, the control vector and disturbance vector will be combined as one input vector, $\underline{u}$. The high order system will be represented as

$$\begin{pmatrix} \dot{\underline{x}}_1 \\ \dot{\underline{x}}_2 \end{pmatrix} = \begin{pmatrix} \underline{\underline{A}}_1 & \underline{\underline{A}}_2 \\ \underline{\underline{A}}_3 & \underline{\underline{A}}_4 \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} + \begin{pmatrix} \underline{\underline{B}}_1 \\ \underline{\underline{B}}_2 \end{pmatrix} \underline{u} \tag{C.1}$$

$$\begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} = \begin{pmatrix} \underline{\underline{M}}_1 & \underline{\underline{M}}_2 \\ \underline{\underline{M}}_3 & \underline{\underline{M}}_4 \end{pmatrix} \begin{pmatrix} \underline{z}_1 \\ \underline{z}_2 \end{pmatrix} \tag{C.2}$$

$$\begin{pmatrix} \dot{\underline{z}}_1 \\ \dot{\underline{z}}_2 \end{pmatrix} = \begin{pmatrix} \underline{\underline{J}}_1 & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{J}}_2 \end{pmatrix} \begin{pmatrix} \underline{z}_1 \\ \underline{z}_2 \end{pmatrix} + \begin{pmatrix} \underline{\underline{G}}_1 \\ \underline{\underline{G}}_2 \end{pmatrix} \underline{u} \tag{C.3}$$

while the low order model will be represented as

$$\dot{\underline{x}}_1 = \underline{\underline{A}}_R \, \underline{x}_1 + \underline{\underline{B}}_R \, \underline{u} \tag{C.4}$$

C.1 EQUIVALENCE OF NICHOLSON'S AND DAVISON'S RESULTS

Both Nicholson [12] and Davison [5] present the same reduced order model in slightly different forms. The difference in their results is in the representation of $\underline{\underline{B}}_R$.

Davison represents $\underline{\underline{B}}_R$ as

$$\underline{\underline{B}}_R^D = \underline{\underline{M}}_1 \, \underline{\underline{G}}_1 \tag{C.5}$$

where $\underline{\underline{G}}_1 = (\underline{\underline{V}}_1, \underline{\underline{V}}_2)\ \underline{\underline{B}}$ (C.6)

Nicholson represents $\underline{\underline{B}}_R$ as

$$\underline{\underline{B}}_R^N = \underline{\underline{\bar{M}}}_1\ \underline{\bar{Q}}_A\ \underline{\underline{B}} \tag{C.7}$$

where $\underline{\underline{\bar{M}}}_1$ is a normalized form of $\underline{\underline{M}}_1$, $\underline{Q}_A$ is the first ℓ rows of $\underline{\underline{M}}^{-1}$ as

$$\underline{Q}_A = (\underline{\underline{V}}_1, \underline{\underline{V}}_2) \tag{C.8}$$

and $\underline{\bar{Q}}_A$ is equivalent to $\underline{Q}_A$ with the i^{th} row multiplied by

$$\frac{m_{ii}}{\bar{m}_{ii}} \qquad (i = 1, 2, \ldots, \ell)$$

The scalars m_{ii} and $\bar{m}_{ii}$ are elements of matrices $\underline{\underline{M}}_1$ and $\underline{\underline{\bar{M}}}_1$, respectively.

In order to show that $\underline{\underline{B}}_R^D = \underline{\underline{B}}_R^N$, Equation (C.7) must be expanded in terms of $\underline{\underline{M}}_1$ and $\underline{Q}_A$.

$\underline{\underline{\bar{M}}}_1$ is the normalized form of $\underline{\underline{M}}_1$. That is, each column of $\underline{\underline{\bar{M}}}_1$ is obtained from $\underline{\underline{M}}_1$ by some normalizing multiplication factor. Let this factor be N_j for the j^{th} column so that

$$\bar{m}_{ij} = N_j\ m_{ij} \tag{C.9}$$

Thus, there is an N_j for each column of $\underline{\underline{M}}_1$. Letting

$$\underline{\underline{N}} = \text{diag}\ (N_1, N_2 \ \ldots\ N_\ell) \tag{C.10}$$

matrix $\underline{\underline{\bar{M}}}_1$ can be written as

$$\underline{\underline{\bar{M}}}_1 = \underline{\underline{M}}_1\ \underline{\underline{N}} \tag{C.11}$$

Similarly, $\bar{\underline{Q}}_A$ is obtained from $\underline{Q}_A$ by multiplying the i^{th} row by $m_{ii}/\bar{m}_{ii}$. Defining

$$\underline{S} = \text{diag}\left(\frac{m_{11}}{\bar{m}_{11}}, \frac{m_{22}}{\bar{m}_{22}} \ldots \frac{m_{\ell\ell}}{\bar{m}_{\ell\ell}}\right) \tag{C.12}$$

Matrix $\bar{\underline{Q}}_A$ can be written as

$$\bar{\underline{Q}}_A = \underline{S}\ \underline{Q}_A \tag{C.13}$$

Now, Equation (C.7) can be written as

$$\begin{aligned} B_R^N &= (\underline{M}_1\ \underline{N})\ (\underline{S}\ \underline{Q}_A)\ \underline{B} \\ &= \underline{M}_1\ (\underline{N}\ \underline{S})\ (\underline{V}_1, \underline{V}_2)\ \underline{B} \\ &= \underline{M}_1\ (\underline{N}\ \underline{S})\ \underline{G}_1 \end{aligned} \tag{C.14}$$

Now, $\underline{N}\ \underline{S} = \text{diag}\left(N_1 \frac{m_{11}}{\bar{m}_{11}}, N_2 \frac{m_{22}}{\bar{m}_{22}}, \ldots N_\ell \frac{m_{\ell\ell}}{\bar{m}_{\ell\ell}}\right)$

$= \underline{I}$, using Equation (C.9)

Thus, $\underline{B}_R^N = \underline{M}_1\ \underline{G}_1 = \underline{B}_R^D$ as desired.

C.2 EQUIVALENCE OF RESULTS OF DAVISON, CHIDAMBARA, FOSSARD AND GRAHAM

As mentioned in Chapter Six, the same reduced order model is presented in the literature in four places in slightly revised forms. The result was first presented by Chidambara [3] as his method C1 in April, 1967, and also by Davison [3] as a revision to his original method in the same correspondence. Davison presented the result

again in [4]. The equivalence of these two was stated by Chidambara [2] in a further correspondence in December, 1967. Graham [9] presented a third form in July, 1968 by a different analysis. Although Graham referred to the correspondence by Chidambara and Davison [3,4] he didn't recognize that his result was identical to theirs. Finally, Fossard [7] presented his form in April, 1970 as an alternative result to Davison's for use when $\underline{\underline{A}}$ and/or $\underline{\underline{A}}_R$ have zero eigenvalues. Although some of the equivalences have been stated before, they all will be shown here due to the simplicity of the proofs. Some basic relations will be used in this work. These are found in Appendix D.

C.2.1 Chidambara's Method (C1)

Chidambara presented his method, referred to as C1, as

$$\dot{\underline{z}}_1 = \underline{\underline{J}}_1 \underline{z}_1 + \underline{\underline{G}}_1 \underline{u} \tag{C.15}$$

$$\underline{x}_1 = \underline{\underline{M}}_1 \underline{z}_1 + \underline{\underline{M}}_2 \underline{z}_2 \tag{C.16}$$

$$\underline{z}_2 = \underline{\underline{F}} \underline{u} = - \underline{\underline{J}}_2^{-1} \underline{\underline{G}}_2 \underline{u} \tag{C.17}$$

From Equations (C.16) and (C.17),

$$\underline{z}_1 = \underline{\underline{M}}_1^{-1} \underline{x}_1 - \underline{\underline{M}}_1^{-1} \underline{\underline{M}}_2 \underline{z}_2$$

$$= \underline{\underline{M}}_1^{-1} \underline{x}_1 + \underline{\underline{M}}_1^{-1} \underline{\underline{M}}_2 \underline{\underline{J}}_2^{-1} \underline{\underline{G}}_2 \underline{u} \tag{C.18}$$

Thus, from Equation (C.16), using Equations (C.15), (C.17), and (C.18), we can write,

$$\dot{\underline{x}}_1 = \underline{M}_1 \dot{\underline{z}}_1 + \underline{M}_2 \dot{\underline{z}}_2$$

$$= \underline{M}_1 \left[\underline{J}_1 \underline{z}_1 + \underline{G}_1 \underline{u}\right] + \underline{M}_2 \left(-\underline{J}_2^{-1} \underline{G}_2 \dot{\underline{u}}\right)$$

$$= \underline{M}_1 \underline{J}_1 \left(\underline{M}_1^{-1} \underline{x}_1 + \underline{M}_1^{-1} \underline{M}_2 \underline{J}_2^{-1} \underline{G}_2 \underline{u}\right) + \underline{M}_1 \underline{G}_1 \underline{u} - \underline{M}_2 \underline{J}_2^{-1} \underline{G}_2 \dot{\underline{u}}$$

$$= \underline{M}_1 \underline{J}_1 \underline{M}_1^{-1} \underline{x}_1 + \underline{M}_1 \left(\underline{G}_1 + \underline{J}_1 \underline{M}_1^{-1} \underline{M}_2 \underline{J}_2^{-1} \underline{G}_2\right) \underline{u} - \underline{M}_2 \underline{J}_2^{-1} \underline{G}_2 \dot{\underline{u}}$$

(C.19)

C.2.2 Davison's Modified Method (DM)

Davison presented his revision to his earlier method as

$$\dot{\underline{x}}_\ell = \left(\underline{A}_1 + \underline{A}_2 \underline{M}_3 \underline{M}_1^{-1}\right) \underline{x}_\ell + \underline{M}_1 \underline{G}_1 \underline{u} \qquad \text{(C.20)}$$

$$\underline{x}_1 = \underline{x}_\ell + \left\{ \underline{A}_R^{-1} \underline{B}_R - \left[\underline{A}^{-1} \underline{B}\right]_\ell \right\} \underline{u} \qquad \text{(C.21)}$$

where $\left[\underline{A}^{-1} \underline{B}\right]_\ell$ are the first ℓ rows of $\underline{A}^{-1} \underline{B}$

First, using Equations (D.1) and (D.2), $\underline{A}_R$ can be written as

$$\underline{A}_R = \underline{A}_1 + \underline{A}_2 \underline{M}_3 \underline{M}_1^{-1}$$

$$= \left(\underline{M}_1 \underline{J}_1 \underline{V}_1 + \underline{M}_2 \underline{J}_2 \underline{V}_3\right) + \left(\underline{M}_1 \underline{J}_1 \underline{V}_2 + \underline{M}_2 \underline{J}_2 \underline{V}_4\right) \underline{M}_3 \underline{M}_1^{-1}$$

$$= \underline{M}_1 \underline{J}_1 \left(\underline{V}_1 + \underline{V}_2 \underline{M}_3 \underline{M}_1^{-1}\right) + \underline{M}_2 \underline{J}_2 \left(\underline{V}_3 + \underline{V}_4 \underline{M}_3 \underline{M}_1^{-1}\right)$$

Now, using Equations (D.7) and (D.19)

$$\underline{A}_R = \underline{M}_1 \underline{J}_1 \left(\underline{V}_1 - \underline{V}_2 \underline{V}_4^{-1} \underline{V}_3\right) + \underline{M}_2 \underline{J}_2 \left(\underline{V}_3 + \underline{V}_4 \underline{M}_3 \underline{M}_1^{-1}\right)$$

$$= \underline{M}_1 \underline{J}_1 \underline{M}_1^{-1} \qquad \text{(C.22)}$$

Now, from Equations (C.20), (C.21), and (C.22)

$$\dot{\underline{x}}_1 = \dot{\underline{x}}_\ell + \left\{ \underline{\underline{A}}_R^{-1}\, \underline{\underline{B}}_R - \left[\underline{\underline{A}}^{-1}\, \underline{\underline{B}} \right]_\ell \right\} \dot{\underline{u}}$$

$$= \underline{\underline{M}}_1\, \underline{\underline{J}}_1\, \underline{\underline{M}}_1^{-1}\, \underline{x}_\ell + \underline{\underline{M}}_1\, \underline{\underline{G}}_1\, \underline{u} + \left\{ \underline{\underline{A}}_R^{-1}\, \underline{\underline{B}}_R - \left[\underline{\underline{A}}^{-1}\, \underline{\underline{B}} \right]_\ell \right\} \dot{\underline{u}}$$

$$= \underline{\underline{M}}_1\, \underline{\underline{J}}_1\, \underline{\underline{M}}_1^{-1} \left\{ \underline{x}_1 - \left\{ \underline{\underline{A}}_R^{-1}\, \underline{\underline{B}}_R - \left[\underline{\underline{A}}^{-1}\, \underline{\underline{B}} \right]_\ell \right\} \underline{u} \right\} + \underline{\underline{M}}_1\, \underline{\underline{G}}_1\, \underline{u} + \left\{ \underline{\underline{A}}_R^{-1}\, \underline{\underline{B}}_R - \left[\underline{\underline{A}}^{-1}\, \underline{\underline{B}} \right]_\ell \right\} \dot{\underline{u}}$$

$$= \underline{\underline{M}}_1\, \underline{\underline{J}}_1\, \underline{\underline{M}}_1^{-1}\, \underline{x}_1 + \underline{\underline{M}}_1 \left[\underline{\underline{G}}_1 - \underline{\underline{J}}_1\, \underline{\underline{M}}_1^{-1} \left\{ \underline{\underline{A}}_R^{-1}\, \underline{\underline{B}}_R - \left[\underline{\underline{A}}^{-1}\, \underline{\underline{B}} \right]_\ell \right\} \right] \underline{u}$$

$$+ \left\{ \underline{\underline{A}}_R^{-1}\, \underline{\underline{B}}_R - \left[\underline{\underline{A}}^{-1}\, \underline{\underline{B}} \right]_\ell \right\} \dot{\underline{u}} \qquad \text{(C.23)}$$

Now, using Appendix D.11, consider the term,

$$\left\{ \underline{\underline{A}}_R^{-1}\, \underline{\underline{B}}_R - \left[\underline{\underline{A}}^{-1}\, \underline{\underline{B}} \right]_\ell \right\}$$

$$= \left(\underline{\underline{M}}_1\, \underline{\underline{J}}_1^{-1}\, \underline{\underline{M}}_1^{-1} \right) \left(\underline{\underline{M}}_1\, \underline{\underline{G}}_1 \right) - \underline{\underline{M}}_1\, \underline{\underline{J}}_1^{-1}\, \underline{\underline{G}}_1 - \underline{\underline{M}}_2\, \underline{\underline{J}}_2^{-1}\, \underline{\underline{G}}_2$$

$$= \underline{\underline{M}}_1\, \underline{\underline{J}}_1^{-1}\, \underline{\underline{G}}_1 - \underline{\underline{M}}_1\, \underline{\underline{J}}_1^{-1}\, \underline{\underline{G}}_1 - \underline{\underline{M}}_2\, \underline{\underline{J}}_2^{-1}\, \underline{\underline{G}}_2$$

$$= - \underline{\underline{M}}_2\, \underline{\underline{J}}_2^{-1}\, \underline{\underline{G}}_2 \qquad \text{(C.24)}$$

Thus, the reduced order model can be written from Equation (C.23) as

$$\dot{\underline{x}}_1 = \underline{\underline{M}}_1\, \underline{\underline{J}}_1\, \underline{\underline{M}}_1^{-1}\, \underline{x}_1 + \underline{\underline{M}}_1 \left(\underline{\underline{G}}_1 + \underline{\underline{J}}_1\, \underline{\underline{M}}_1^{-1}\, \underline{\underline{M}}_2\, \underline{\underline{J}}_2^{-1}\, \underline{\underline{G}}_2 \right) \underline{u} - \underline{\underline{M}}_2\, \underline{\underline{J}}_2^{-1}\, \underline{\underline{G}}_2\, \dot{\underline{u}} \qquad \text{(C.25)}$$

Equation (C.25) is equivalent to Equation (C.19) and so Chidambara's C1 method and Davison's DM method are equivalent.

C.2.3 Fossard's Revision

Fossard presented the revision to Davison's method as

$$\dot{\underline{x}}_\ell = (\underline{\underline{A}}_1 + \underline{\underline{A}}_2 \underline{\underline{M}}_3 \underline{\underline{M}}_1^{-1}) \underline{x}_\ell + \underline{\underline{M}}_1 \underline{\underline{G}}_1 \underline{u} \qquad (C.26)$$

$$\underline{x}_1 = \underline{x}_\ell - \underline{\underline{M}}_2 \underline{\underline{J}}_2^{-1} \underline{\underline{G}}_2 \underline{u} \qquad (C.27)$$

In Section C.2.2, Equation (C.24) showed that

$$- \underline{\underline{M}}_2 \underline{\underline{J}}_2^{-1} \underline{\underline{G}}_2 = \left\{ \underline{\underline{A}}_R^{-1} \underline{\underline{B}}_R - \left[\underline{\underline{A}}^{-1} \underline{\underline{B}} \right]_\ell \right\} \qquad (C.28)$$

so that the correction applied by Fossard is identical to that applied by Davison, and so this result is equivalent to the previous two results, since Equations (C.26) and (C.27) can be written as

$$\dot{\underline{x}}_\ell = (\underline{\underline{A}}_1 + \underline{\underline{A}}_2 \underline{\underline{M}}_3 \underline{\underline{M}}_1^{-1}) \underline{x}_R + \underline{\underline{M}}_1 \underline{\underline{G}}_1 \underline{u} \qquad (C.29)$$

$$\underline{x}_1 = \underline{x}_\ell + \left\{ \underline{\underline{A}}_R^{-1} \underline{\underline{B}}_R - \left[\underline{\underline{A}}^{-1} \underline{\underline{B}} \right]_\ell \right\} \underline{u} \qquad (C.30)$$

which is identical to Equations (C.20) and (C.21).

C.2.4 Graham's Result

Graham presented his result as

$$\dot{\underline{x}}_1 = \underline{\underline{M}}_1 \underline{\underline{J}}_1 \underline{\underline{M}}_1^{-1} \underline{x}_1 + (\underline{\underline{B}}_1 - \underline{\underline{A}}_2 \underline{\underline{V}}_4^{-1} \underline{\underline{J}}_2^{-1} \underline{\underline{G}}_2) \underline{u} - \underline{\underline{M}}_2 \underline{\underline{J}}_2^{-1} \underline{\underline{G}}_2 \dot{\underline{u}} \qquad (C.31)$$

This is already in the general form of Equations (C.19) and (C.25) except for the representation of $\underline{\underline{B}}_R$. $\underline{\underline{B}}_R$ from Equation (C.19) can be arranged into the form as presented above. Thus, from Equation (C.19) using Equation (D.10),

$$\underline{B}_R = \underline{M}_1 \left(\underline{G}_1 + \underline{J}_1 \underline{M}_1^{-1} \underline{M}_2 \underline{J}_2^{-1} \underline{G}_2\right)$$

$$= \underline{M}_1 \underline{G}_1 + \underline{M}_1 \underline{J}_1 \left(-\underline{V}_2 \underline{V}_4^{-1}\right) \underline{J}_2^{-1} \underline{G}_2$$

$$= \underline{M}_1 \left(\underline{V}_1 \underline{B}_1 + \underline{V}_2 \underline{B}_2\right) - \underline{M}_1 \underline{J}_1 \underline{V}_2 \underline{V}_4^{-1} \underline{J}_2^{-1} \underline{G}_2$$

Now, using Equations (D.2) and (D.9)

$$\underline{B}_R = \left(\underline{I} - \underline{M}_2 \underline{V}_3\right) \underline{B}_1 + \underline{M}_1 \underline{V}_2 \underline{B}_2 - \left(\underline{A}_2 - \underline{M}_2 \underline{J}_2 \underline{V}_4\right) \underline{V}_4^{-1} \underline{J}_2^{-1} \underline{G}_2$$

$$= \underline{B}_1 - \underline{M}_2 \underline{V}_3 \underline{B}_1 + \underline{M}_1 \underline{V}_2 \underline{B}_2 - \underline{A}_2 \underline{V}_4^{-1} \underline{J}_2^{-1} \underline{G}_2 + \underline{M}_2 \underline{G}_2$$

$$= \underline{B}_1 - \underline{A}_2 \underline{V}_4^{-1} \underline{J}_2^{-1} \underline{G}_2 - \underline{M}_2 \underline{V}_3 \underline{B}_1 + \underline{M}_1 \underline{V}_2 \underline{B}_2 + \underline{M}_2 \underline{V}_3 \underline{B}_1 + \underline{M}_2 \underline{V}_4 \underline{B}_2$$

$$= \underline{B}_1 - \underline{A}_2 \underline{V}_4^{-1} \underline{J}_2^{-1} \underline{G}_2 + \left(\underline{M}_1 \underline{V}_2 + \underline{M}_2 \underline{V}_4\right) \underline{B}_2$$

Finally, using Equation (D.10)

$$\underline{B}_R = \underline{B}_1 - \underline{A}_2 \underline{V}_4^{-1} \underline{J}_2^{-1} \underline{G}_2$$

which is the form of $\underline{B}_R$ above in Equation (C.31).

Thus, Graham's result can also be written as

$$\dot{\underline{x}}_1 = \underline{M}_1 \underline{J}_1 \underline{M}_1^{-1} \underline{x}_1 + \underline{M}_1 \left(\underline{G}_1 + \underline{J}_1 \underline{M}_1^{-1} \underline{M}_2 \underline{J}_2^{-1} \underline{G}_2\right) \underline{u} - \underline{M}_2 \underline{J}_2^{-1} \underline{G}_2 \dot{\underline{u}} \qquad \text{(C.32)}$$

which is equivalent to the other three results in Appendix C.2.

C.3 EQUIVALENCE OF RESULTS OF MARSHALL AND CHIDAMBARA

The results presented by Marshall and by Chidambara as method C2 are identical. Marshall presents $\underline{A}_R$ as

$$\underline{A}_R = \underline{M}_1 \underline{J}_1 \underline{M}_1^{-1}$$

while Chidambara presents it as

$$\underline{A}_R = (\underline{A}_1 + \underline{A}_2 \underline{M}_3 \underline{M}_1^{-1})$$

These two forms of $\underline{A}_R$ are shown to be equivalent in Appendix C.2.2.

The form of $\underline{B}_R$ presented by Marshall is

$$\underline{B}_R = (\underline{B}_1 - \underline{A}_2 \underline{V}_4^{-1} \underline{J}_2^{-1} (\underline{V}_3 \underline{B}_1 + \underline{V}_4 \underline{B}_2)) \qquad (C.33)$$

while that presented by Chidambara is

$$\underline{B}_R = \underline{B}_1 + (\underline{A}_2 \underline{M}_4 - \underline{A}_2 \underline{M}_3 \underline{M}_1^{-1} \underline{M}_2) \underline{F} \qquad (C.34)$$

where $\underline{F} = -\underline{J}_2^{-1} \underline{G}_2 \qquad (C.35)$

Equations (C.34) and (C.35) can be combined, and rearranged as follows:

$$\underline{B}_R = \underline{B}_1 - \underline{A}_2 (\underline{M}_4 - \underline{M}_3 \underline{M}_1^{-1} \underline{M}_2) \underline{J}_2^{-1} \underline{G}_2$$

$$= \underline{B}_1 - \underline{A}_2 (\underline{M}_4 - \underline{M}_3 \underline{M}_1^{-1} \underline{M}_2) \underline{J}_2^{-1} (\underline{V}_3 \underline{B}_1 + \underline{V}_4 \underline{B}_2)$$

Using Equations (D.7) and (D.8)

$$\underline{V}_4^{-1} = (\underline{M}_4 - \underline{M}_3 \underline{M}_1^{-1} \underline{M}_2)$$

so that

$$\underline{B}_R = \underline{B}_1 - \underline{A}_2 \underline{V}_4^{-1} \underline{J}_2^{-1} (\underline{V}_3 \underline{B}_1 + \underline{V}_4 \underline{B}_2)$$

which is identical to Equation (C.33). Thus, Marshall's result is identical to Chidambara's C2 result.

APPENDIX D

MATHEMATICAL EQUIVALENCES

D.1 THEOREM

Gantmacher [8, p. 98] presents a theorem that is used in this thesis. It states:

If $\underline{A}$ and $\underline{B}$ are similar and $\underline{W}$ transforms $\underline{A}$ into $\underline{B}$,

$$\underline{B} = \underline{W}^{-1} \underline{A} \underline{W},$$

then the matrices $f(\underline{A})$ and $f(\underline{B})$ are also similar and $\underline{W}$ transforms $f(\underline{A})$ into $f(\underline{B})$,

$$f(\underline{B}) = \underline{W}^{-1} f(\underline{A}) \underline{W}$$

For the continuous-time system of Equation (3.36), from Equation (3.39),

$$\underline{J} = \underline{M}^{-1} \underline{A} \underline{M}$$

From the above theorem, and the definition of $\underline{\Phi}$ as $\underline{\Phi} = \exp(\underline{A} T)$, it follows that,

$$\exp(\underline{J}T) = \underline{M}^{-1} \exp(\underline{A}T) \underline{M} = \underline{M}^{-1} \underline{\Phi} \underline{M} = \underline{\alpha}$$

Thus, the same transformation matrix, $\underline{M}$, will transform both the discrete-time and the continuous-time systems into their corresponding Jordan canonical forms, and $\underline{M}$ can be calculated using either the discrete-time system or the continuous-time system.

D.2 PARTITIONS OF $\underline{A}$

Starting with $\underline{A} = \underline{M}\,\underline{J}\,\underline{V}$, it follows that

$$\begin{pmatrix} \underline{A}_1 & \underline{A}_2 \\ \underline{A}_3 & \underline{A}_4 \end{pmatrix} = \begin{pmatrix} \underline{M}_1\,\underline{J}_1\,\underline{V}_1 + \underline{M}_2\,\underline{J}_2\,\underline{V}_3 & \underline{M}_1\,\underline{J}_1\,\underline{V}_2 + \underline{M}_2\,\underline{J}_2\,\underline{V}_4 \\ \underline{M}_3\,\underline{J}_1\,\underline{V}_1 + \underline{M}_4\,\underline{J}_2\,\underline{V}_3 & \underline{M}_3\,\underline{J}_1\,\underline{V}_2 + \underline{M}_4\,\underline{J}_2\,\underline{V}_4 \end{pmatrix}$$

Consequently,

$$\underline{A}_1 = \underline{M}_1\,\underline{J}_1\,\underline{V}_1 + \underline{M}_2\,\underline{J}_2\,\underline{V}_3 = (\underline{M}_1, \underline{M}_2)\,\underline{J}\begin{pmatrix} \underline{V}_1 \\ \underline{V}_3 \end{pmatrix} \qquad (D.1)$$

$$\underline{A}_2 = \underline{M}_1\,\underline{J}_1\,\underline{V}_2 + \underline{M}_2\,\underline{J}_2\,\underline{V}_4 = (\underline{M}_1, \underline{M}_2)\,\underline{J}\begin{pmatrix} \underline{V}_2 \\ \underline{V}_4 \end{pmatrix} \qquad (D.2)$$

D.3 PARTITIONS OF $\underline{\Phi}$

Following from $\underline{A} = \underline{M}\,\underline{J}\,\underline{V}$, using the theorem in section D.1, $\underline{\Phi} = \underline{M}\,e^{\underline{J}T}\,\underline{V}$. Thus,

$$\begin{pmatrix} \underline{\Phi}_1 & \underline{\Phi}_2 \\ \underline{\Phi}_3 & \underline{\Phi}_4 \end{pmatrix} = \begin{pmatrix} \underline{M}_1\,e^{\underline{J}_1 T}\,\underline{V}_1 + \underline{M}_2\,e^{\underline{J}_2 T}\,\underline{V}_3 & \underline{M}_1\,e^{\underline{J}_1 T}\,\underline{V}_2 + \underline{M}_2\,e^{\underline{J}_2 T}\,\underline{V}_4 \\ \underline{M}_3\,e^{\underline{J}_1 T}\,\underline{V}_1 + \underline{M}_4\,e^{\underline{J}_2 T}\,\underline{V}_3 & \underline{M}_3\,e^{\underline{J}_1 T}\,\underline{V}_2 + \underline{M}_4\,e^{\underline{J}_2 T}\,\underline{V}_4 \end{pmatrix}$$

Consequently,

$$\underline{\Phi}_1 = \underline{M}_1\,e^{\underline{J}_1 T}\,\underline{V}_1 + \underline{M}_2\,e^{\underline{J}_2 T}\,\underline{V}_3 = (\underline{M}_1, \underline{M}_2)\,e^{\underline{J}T}\begin{pmatrix} \underline{V}_1 \\ \underline{V}_3 \end{pmatrix} \qquad (D.3)$$

$$\underline{\Phi}_2 = \underline{M}_1\,e^{\underline{J}_1 T}\,\underline{V}_2 + \underline{M}_2\,e^{\underline{J}_2 T}\,\underline{V}_4 = (\underline{M}_1, \underline{M}_2)\,e^{\underline{J}T}\begin{pmatrix} \underline{V}_2 \\ \underline{V}_4 \end{pmatrix} \qquad (D.4)$$

D.4 EXPANSION OF $\underline{VM} = \underline{I}$

By definition of $\underline{V} = \underline{M}^{-1}$, it follows that $\underline{VM} = \underline{I}$ and

$$\begin{pmatrix} \underline{V}_1 \underline{M}_1 + \underline{V}_2 \underline{M}_3 & \underline{V}_1 \underline{M}_2 + \underline{V}_2 \underline{M}_4 \\ \underline{V}_3 \underline{M}_1 + \underline{V}_4 \underline{M}_3 & \underline{V}_3 \underline{M}_2 + \underline{V}_4 \underline{M}_4 \end{pmatrix} = \begin{pmatrix} \underline{I} & \underline{0} \\ \underline{0} & \underline{I} \end{pmatrix}$$

which implies

$$\underline{V}_1 \underline{M}_1 + \underline{V}_2 \underline{M}_3 = \underline{I} \qquad \text{(D.5)}$$

$$\underline{V}_1 \underline{M}_2 + \underline{V}_2 \underline{M}_4 = \underline{0} \qquad \text{(D.6)}$$

$$\underline{V}_3 \underline{M}_1 + \underline{V}_4 \underline{M}_3 = \underline{0} \qquad \text{(D.7)}$$

$$\underline{V}_3 \underline{M}_2 + \underline{V}_4 \underline{M}_4 = \underline{I} \qquad \text{(D.8)}$$

D.5 EXPANSION OF $\underline{\underline{M}}\underline{\underline{V}} = \underline{I}$

In a manner similar to Appendix D.4, $\underline{\underline{M}}\underline{\underline{V}} = \underline{I}$ and

$$\begin{pmatrix} \underline{M}_1 \underline{V}_1 + \underline{M}_2 \underline{V}_3 & \underline{M}_1 \underline{V}_2 + \underline{M}_2 \underline{V}_4 \\ \underline{M}_3 \underline{V}_1 + \underline{M}_4 \underline{V}_3 & \underline{M}_3 \underline{V}_2 + \underline{M}_4 \underline{V}_4 \end{pmatrix} = \begin{pmatrix} \underline{I} & \underline{0} \\ \underline{0} & \underline{I} \end{pmatrix}$$

and

$$\underline{M}_1 \underline{V}_1 + \underline{M}_2 \underline{V}_3 = \underline{I} \qquad \text{(D.9)}$$

$$\underline{M}_1 \underline{V}_2 + \underline{M}_2 \underline{V}_4 = \underline{0} \qquad \text{(D.10)}$$

$$\underline{M}_3 \underline{V}_1 + \underline{M}_4 \underline{V}_3 = \underline{0} \qquad \text{(D.11)}$$

$$\underline{M}_3 \underline{V}_2 + \underline{M}_4 \underline{V}_4 = \underline{I} \qquad \text{(D.12)}$$

D.6 PARTITIONS OF $\underline{\Delta}$

Starting with $\underline{\Delta} = \int_0^T e^{\underline{A}(T-\tau)}\ \underline{B}\ d\tau$ and the theorem in Appendix D.1, it follows that

$$\underline{\Delta} = \begin{pmatrix} \underline{M}_1 & \underline{M}_2 \\ \underline{M}_3 & \underline{M}_4 \end{pmatrix} \int_0^T \begin{bmatrix} e^{\underline{J}_1(T-\tau)} & \underline{0} \\ \underline{0} & e^{\underline{J}_2(T-\tau)} \end{bmatrix} \begin{pmatrix} \underline{V}_1 & \underline{V}_2 \\ \underline{V}_3 & \underline{V}_4 \end{pmatrix} \begin{pmatrix} \underline{B}_1 \\ \underline{B}_2 \end{pmatrix} d\tau$$

and therefore

$$\underline{\Delta}_1 = (\underline{M}_1, \underline{M}_2) \int_0^T \begin{bmatrix} e^{\underline{J}_1(T-\tau)} & \underline{0} \\ \underline{0} & e^{\underline{J}_2(T-\tau)} \end{bmatrix} \begin{bmatrix} \underline{V}_1 & \underline{V}_2 \\ \underline{V}_3 & \underline{V}_4 \end{bmatrix} \begin{bmatrix} \underline{B}_1 \\ \underline{B}_2 \end{bmatrix} d\tau \qquad (D.13)$$

D.7 DEFINITIONS OF $\underline{\alpha}_2$ AND $\underline{\delta}_2$

Since $\underline{\alpha} = e^{\underline{J}T}$, then $\underline{\alpha}_2 = e^{\underline{J}_2 T}$ (D.14)

Since $\underline{\delta} = \underline{V}\ \underline{\Delta}$, then $\underline{\delta}_2 = (\underline{V}_3, \underline{V}_4)\ \underline{\Delta}$ (D.15)

D.8 RELATIONS FOR $\underline{V}_1$, $\underline{V}_2$, $\underline{M}_2$, $\underline{M}_1^{-1}$

From Equation (D.9), $\underline{V}_1 = \underline{M}_1^{-1}\ (\underline{I} - \underline{M}_2\ \underline{V}_3)$ (D.16)

From Equation (D.10), $\underline{V}_2 = -\ \underline{M}_1^{-1}\ \underline{M}_2\ \underline{V}_4$ (D.17)

Rearranging Equation (D.2), $\underline{M}_2$ can be calculated as

$$\underline{M}_2 = \underline{A}_2\ \underline{V}_4^{-1}\ \underline{J}_2^{-1} - \underline{M}_1\ \underline{J}_1\ \underline{V}_2\ \underline{V}_4^{-1}\ \underline{J}_2^{-1} \qquad (D.18)$$

Eliminating $\underline{M}_2$ from Equations (D.9) and (D.10) gives

$$\underline{M}_1^{-1} = (\underline{V}_1 - \underline{V}_2 \underline{V}_4^{-1} \underline{V}_3) \qquad (D.19)$$

D.9 INTEGRAL EVALUATION NO. 1

The following integral can be evaluated as shown here

$$\int_0^T e^{\underline{W}(T-\tau)} d\tau = e^{\underline{W}T} \int_0^T e^{-\underline{W}\tau} d\tau$$

$$\int_0^T e^{-\underline{W}\tau} d\tau = \int_0^T \left(\underline{I} - \underline{W}\tau + \frac{\underline{W}^2\tau^2}{2} - \frac{\underline{W}^3\tau^3}{3!} + \ldots \right) d\tau$$

$$= \left\{ T\underline{I} - \frac{\underline{W}T^2}{2} + \frac{\underline{W}^2T^3}{3!} - \frac{\underline{W}^3T^4}{4!} + \ldots \right\}$$

$$= \left\{ \underline{W}T - \frac{\underline{W}^2T^2}{2!} + \frac{\underline{W}^3T^3}{3!} - \frac{\underline{W}^4T^4}{4!} + \ldots \right\} \underline{W}^{-1}$$

$$= \left\{ \underline{I} - \left(\underline{I} - \underline{W}T + \frac{\underline{W}^2T^2}{2} - \frac{\underline{W}^3T^3}{3!} + \ldots \right) \right\} \underline{W}^{-1}$$

$$= \left(\underline{I} - e^{-\underline{W}T} \right) \underline{W}^{-1}$$

$$\therefore \int_0^T e^{\underline{W}(T-\tau)} d\tau = \left(e^{\underline{W}T} - \underline{I} \right) \underline{W}^{-1}$$

D.10 INTEGRAL EVALUATION NO. 2

$$\int_0^T e^{\underline{W}(T-\tau)} \underline{W} d\tau = e^{\underline{W}T} \int_0^T e^{-\underline{W}\tau} \underline{W} d\tau$$

$$\int_0^T e^{-\underline{W}\tau} \underline{W} d\tau = \int_0^T \left(\underline{I} - \underline{W}\tau + \frac{\underline{W}^2\tau^2}{2} - \frac{\underline{W}^3\tau^3}{3!} + \ldots \right) \underline{W} d\tau$$

$$= \left(\underline{W}T - \frac{\underline{W}^2T^2}{2} + \frac{\underline{W}^3T^3}{3!} - \frac{\underline{W}^4T^4}{4!} + \ldots \right)$$

$$= \left(\underline{I} - e^{-\underline{W}T}\right)$$

$$\therefore \int_0^T e^{\underline{W}(T-\tau)} \underline{W} d\tau = \left(e^{\underline{W}T} - \underline{I}\right)$$

D.11 EXPRESSION FOR $\left[\underline{A}^{-1}\ \underline{B}\right]_\ell$

Starting with the equality $\underline{A} = \underline{M}\ \underline{J}\ \underline{V}$,

$$\underline{A}^{-1}\ \underline{B} = \underline{M}\ \underline{J}^{-1}\ \underline{V}\ \underline{B}$$

$$= \begin{bmatrix} \underline{M}_1 & \underline{M}_2 \\ \underline{M}_3 & \underline{M}_4 \end{bmatrix} \begin{bmatrix} \underline{J}_1^{-1} & \underline{0} \\ \underline{0} & \underline{J}_2^{-1} \end{bmatrix} \begin{bmatrix} \underline{V}_1 & \underline{V}_2 \\ \underline{V}_3 & \underline{V}_4 \end{bmatrix} \begin{bmatrix} \underline{B}_1 \\ \underline{B}_2 \end{bmatrix}$$

Now, $\left[\underline{A}^{-1}\ \underline{B}\right]_\ell$ is the first ℓ rows of $\underline{A}^{-1}\ \underline{B}$ as

$$\left[\underline{A}^{-1}\ \underline{B}\right]_\ell = (\underline{M}_1,\ \underline{M}_2) \begin{pmatrix} \underline{J}_1^{-1} & \underline{0} \\ \underline{0} & \underline{J}_2^{-1} \end{pmatrix} \begin{pmatrix} \underline{G}_1 \\ \underline{G}_2 \end{pmatrix}$$

$$= \underline{M}_1\ \underline{J}_1^{-1}\ \underline{G}_1 + \underline{M}_2\ \underline{J}_2^{-1}\ \underline{G}_2$$

D.12 CONDITIONS FOR WHICH EXP $(\underline{A}\ \underline{B}\ \underline{C})$ = $\underline{A}$ EXP $(\underline{B})\ \underline{C}$

Consider the power series expansion of exp $(\underline{A}\ \underline{B}\ \underline{C})$ as

$$\exp\ (\underline{A}\ \underline{B}\ \underline{C}) = \underline{I} + \underline{A}\ \underline{B}\ \underline{C} + \frac{(\underline{A}\ \underline{B}\ \underline{C})\ (\underline{A}\ \underline{B}\ \underline{C})}{2!} + \frac{(\underline{A}\ \underline{B}\ \underline{C})\ (\underline{A}\ \underline{B}\ \underline{C})\ (\underline{A}\ \underline{B}\ \underline{C})}{3!} + \ldots$$

if $\underline{C} = \underline{A}^{-1}$ this can be written as

$$\exp\ (\underline{A}\ \underline{B}\ \underline{C}) = \underline{A}\ (\underline{I})\ \underline{C} + \underline{A}\ \underline{B}\ \underline{C} + \frac{\underline{A}\ \underline{B}^2\ \underline{C}}{2!} + \frac{\underline{A}\ \underline{B}^3\ \underline{C}}{3!} + \ldots$$

$$= \underline{A}\left(\underline{I} + \underline{B} + \frac{\underline{B}^2}{2!} + \frac{\underline{B}^3}{3!} + \ldots\right)\underline{C}$$

$$= \underline{A}\ \exp\ (\underline{B})\ \underline{C}$$

Thus, $\exp\ (\underline{A}\ \underline{B}\ \underline{C}) = \underline{A}\ \exp\ (\underline{B})\ \underline{C}$ if $\underline{A} = \underline{C}^{-1}$

APPENDIX E

LEAST SQUARES APPROACH TO CONTROL LAW DESIGN

E.1 MINIMIZATION WITH RESPECT TO THE CONTROL LAW

The desired form of the reduced order feedback control law is

$$\underline{u} = \underline{\underline{K}}_R^{FB}\, \underline{x}_1 \qquad \text{or} \qquad \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} \underline{K}_{R_1}^{FB} \\ \vdots \\ \underline{K}_{R_m}^{FB} \end{bmatrix} \underline{x}_1 \tag{E.1}$$

Consider only the first of these equations, that for the scalar u_1 as

$$u_1 = \underline{K}_{R_1}^{FB}\, \underline{x}_1 \tag{E.2}$$

$$= \underline{x}_1^T \left(\underline{K}_{R_1}^{FB}\right)^T \tag{E.3}$$

Rewrite Equation (E.3) for the k data points used in the least squares solution

$$\begin{bmatrix} u_1(1) \\ u_1(2) \\ \vdots \\ u_1(k) \end{bmatrix} = \begin{bmatrix} \underline{x}_1^T(1) \\ \underline{x}_1^T(2) \\ \vdots \\ \underline{x}_1^T(k) \end{bmatrix} \left(\underline{K}_{R_1}^{FB}\right)^T \tag{E.4}$$

or

$$\underline{w}_1 = \underline{\underline{Z}} \left(\underline{K}_{R_1}^{FB}\right)^T \tag{E.5}$$

We want to calc $\left(\underline{K}_{R_1}^{FB}\right)$ in order to minimize the sum of the squares between the left side and the right side of Equation (E.5) as

$$S^2 = \left(\underline{w}_1 - \underline{\underline{Z}}\left(\underline{K}_{R_1}^{FB}\right)^T\right)^T \left(\underline{w}_1 - \underline{\underline{Z}}\left(\underline{K}_{R_1}^{FB}\right)^T\right) \qquad \text{(E.6)}$$

$$= \underline{w}_1^T \underline{w}_1 - 2\,\underline{K}_{R_1}^{FB}\,\underline{\underline{Z}}^T \underline{w}_1 + \left(\underline{K}_{R_1}^{FB}\right)\,\underline{\underline{Z}}^T \underline{\underline{Z}}\left(\underline{K}_{R_1}^{FB}\right)^T$$

Minimize S^2 to give

$$\frac{\partial S^2}{\partial\left(\underline{K}_{R_1}^{FB}\right)^T} = -2\,\underline{\underline{Z}}^T \underline{w}_1 + 2\,\underline{\underline{Z}}^T \underline{\underline{Z}}\left(\underline{K}_{R_1}^{FB}\right)^T = \underline{0} \qquad \text{(E.7)}$$

Thus,

$$\left(\underline{K}_{R_1}^{FB}\right)^T = \left(\underline{\underline{Z}}^T \underline{\underline{Z}}\right)^{-1} \underline{\underline{Z}}^T \underline{w}_1 \qquad \text{(E.8)}$$

The steps from Equations (E.2) to (E.8) can be done for each of the equations represented in Equation (E.1) and the results will be

$$\left.\begin{aligned} \left(\underline{K}_{R_2}^{FB}\right)^T &= \left(\underline{\underline{Z}}^T \underline{\underline{Z}}\right)^{-1} \underline{\underline{Z}}^T \underline{w}_2 \\ \left(\underline{K}_{R_3}^{FB}\right)^T &= \left(\underline{\underline{Z}}^T \underline{\underline{Z}}\right)^{-1} \underline{\underline{Z}}^T \underline{w}_3 \end{aligned}\right\} \qquad \text{(E.9)}$$

etc

Thus, Equations (E.8) and (E.9) can be augmented together to give one matrix equation for the resulting control law, as

$$\left(\underline{\underline{K}}_R^{FB}\right)^T = \left(\underline{K}_{R_1}^{FB}, \underline{K}_{R_2}^{FB}, \ldots, \underline{K}_{R_m}^{FB}\right)^T = \left(\underline{\underline{Z}}^T \underline{\underline{Z}}\right)^{-1} \underline{\underline{Z}}^T \left(\underline{w}_1, \underline{w}_2, \ldots \underline{w}_m\right)$$

$$= \left(\underline{\underline{Z}}^T \underline{\underline{Z}}\right)^{-1} \underline{\underline{Z}}^T \underline{\underline{W}} \qquad \text{(E.10)}$$

If the desired reduced order control law is in the form of

$$\underline{u} = \underline{\underline{K}}_R^{FB}\,\underline{x}_1 + \underline{\underline{K}}_R^{FB}\,\underline{d} + \underline{\underline{K}}_R^{SP}\,\underline{y}^{SP} + \underline{\underline{K}}_R^{I}\,\underline{s}, \tag{E.11}$$

it can be rewritten as

$$\underline{u} = \hat{\underline{\underline{K}}}\,\underline{v} \tag{E.12}$$

where

$$\hat{\underline{\underline{K}}} = \left(\underline{\underline{K}}_R^{FB},\ \underline{\underline{K}}_R^{FF},\ \underline{\underline{K}}_R^{SP},\ \underline{\underline{K}}_R^{I}\right) \tag{E.13}$$

and

$$\underline{v} = \begin{bmatrix} \underline{x}_1 \\ \underline{d} \\ \underline{y}^{SP} \\ \underline{s} \end{bmatrix} \tag{E.14}$$

and the procedure is continued as above.

In the evaluation of Equation (E.10), the elements of $\underline{\underline{W}}$ are evaluated using the high order, optimal control law, so that the rows of $\underline{\underline{W}}$ are as

$$\underline{u}(j) = \underline{\underline{K}}_1^{FB}\,\underline{x}_1(j) + \underline{\underline{K}}_2^{FB}\,\underline{x}_2(j) \tag{E.15}$$

or

$$\underline{u}^T(j) = \underline{x}_1^T(j)\left(\underline{\underline{K}}_1^{FB}\right)^T + \underline{x}_2^T(j)\left(\underline{\underline{K}}_2^{FB}\right)^T \tag{E.16}$$

and $\underline{\underline{W}}$ becomes

$$\underline{\underline{W}} = \underline{\underline{Z}}\left(\underline{\underline{K}}_1^{FB}\right)^T + \underline{\underline{Z}}_2\left(\underline{\underline{K}}_2^{FB}\right)^T \tag{E.17}$$

where

$$\underline{Z}_2 = \begin{bmatrix} \underline{x}_2^T(1) \\ \underline{x}_2^T(2) \\ \vdots \\ \underline{x}_2^T(k) \end{bmatrix} \tag{E.18}$$

Now, Equation (E.10) can be written as

$$\left(\underline{K}_R^{FB}\right)^T = (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T \left(\underline{Z}\left(\underline{K}_1^{FB}\right)^T + \underline{Z}_2 \left(\underline{K}_2^{FB}\right)^T\right)$$

$$= \left(\underline{K}_1^{FB}\right)^T + (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T \underline{Z}_2 \left(\underline{K}_2^{FB}\right)^T \tag{E.19}$$

E.2 MINIMIZATION WITH RESPECT TO THE HIGH ORDER CLOSED-LOOP MATRIX

In this second formulation, the desired resulting reduced order control law is still as in Equation (E.1) but the minimization is done with respect to the system's closed loop matrix. The closed loop form is

$$\underline{x}(j+1) = \underline{\Phi}_1 \underline{x}_1(j) + \underline{\Phi}_2 \underline{x}_2(j) + \underline{\Delta} \underline{K}_R^{FB} \underline{x}_1(j)$$

$$= \left(\underline{\Phi}_1 + \underline{\Delta} \underline{K}_R^{FB}\right) \underline{x}_1(j) + \underline{\Phi}_2 \underline{x}_2(j) \tag{E.20}$$

where

$$\underline{\Phi} = (\underline{\Phi}_1, \underline{\Phi}_2) = \begin{bmatrix} \underline{\Phi}_{11}, \underline{\Phi}_{21} \\ \underline{\Phi}_{12}, \underline{\Phi}_{22} \\ \vdots \quad \vdots \\ \underline{\Phi}_{1n}, \underline{\Phi}_{2n} \end{bmatrix} \tag{E.21}$$

Also, $\underline{\Delta} = \begin{bmatrix} \underline{\Delta}_1 \\ \underline{\Delta}_2 \\ \vdots \\ \underline{\Delta}_n \end{bmatrix}$

Consider only the first equation of (E.20), for scalar $x_1(j+1)$ as,

$$x_1(j+1) = \underline{C}_1^T \underline{x}_1(j) + \underline{\Phi}_{21} \underline{x}_2(j)$$

$$= \underline{x}_1^T(j) \underline{C}_1 + \underline{x}_2^T(j) \underline{\Phi}_{21}^T \qquad \text{(E.22)}$$

where

$$\underline{C}_1 = \left(\underline{\Phi}_{11} + \underline{\Delta}_1 \underline{K}_R^{FB} \right)^T$$

Writing Equation (E-22) for k data points gives

$$\left.\begin{aligned} x_1(1) &= \underline{x}_1^T(1) \underline{C}_1 + \underline{x}_2^T(1) \underline{\Phi}_{21}^T \\ x_1(2) &= \underline{x}_1^T(2) \underline{C}_1 + \underline{x}_2^T(2) \underline{\Phi}_{21}^T \\ &\vdots \\ x_1(k) &= \underline{x}_1^T(k) \underline{C}_1 + \underline{x}_2^T(k) \underline{\Phi}_{21}^T \end{aligned}\right\} \text{(E.23)}$$

Lump Equation (E.23) to form one matrix equation as

$$\underline{w}_1 = \underline{Z} \underline{C}_1 + \underline{Z}_2 \underline{\Phi}_{21}^T \qquad \text{(E.24)}$$

We want to calculate $\underline{K}_R^{FB}$ by minimizing the sum of the squares of the difference between the left side and right side of Equation (E.24),

$$S^2 = (\underline{w}_1 - \underline{Z} \underline{C}_1 - \underline{Z}_2 \underline{\Phi}_{21}^T)^T (\underline{w}_1 - \underline{Z} \underline{C}_1 - \underline{Z}_2 \underline{\Phi}_{21}^T)$$

$$= \underline{w}_1^T (\underline{w}_1 - \underline{Z} \underline{C}_1 - \underline{Z}_2 \underline{\Phi}_{21}^T) - \underline{C}_1^T \underline{Z}^T (\underline{w}_1 - \underline{Z} \underline{C}_1 - \underline{Z}_2 \underline{\Phi}_{21}^T)$$

$$- \underline{\Phi}_{21} \underline{Z}_2^T (\underline{w}_1 - \underline{Z} \underline{C}_1 - \underline{Z}_2 \underline{\Phi}_{21}^T)$$

with respect to $\underline{C}_1$ as

$$\frac{\partial s^2}{\partial \underline{C}_1} = - \underline{Z}^T \underline{w}_1 - \underline{Z}^T (\underline{w}_1 - \underline{Z} \underline{C}_1 - \underline{Z}_2 \underline{\Phi}_{21}^T) + \underline{Z}^T \underline{Z} \underline{C}_1 + \underline{Z}^T \underline{Z}_2 \underline{\Phi}_{21}^T = \underline{0}$$

$$\underline{C}_1 = (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T (\underline{w}_1 - \underline{Z}_2 \underline{\Phi}_{21}^T) \qquad \text{(E.25)}$$

The steps from Equations (E.22) to (E.25) can be done for each of the equations in (E.20) to give

$$\left.\begin{aligned} \underline{C}_2 &= (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T (\underline{w}_2 - \underline{Z}_2 \underline{\Phi}_{22}^T) \\ \underline{C}_3 &= (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T (\underline{w}_3 - \underline{Z}_2 \underline{\Phi}_{23}^T) \\ &\vdots \\ \underline{C}_n &= (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T (\underline{w}_n - \underline{Z}_2 \underline{\Phi}_{2n}^T) \end{aligned}\right\} \text{(E.26)}$$

Thus, Equations (E.26) and (E.25) can be augmented together to form the resulting suboptimal closed loop matrix as

$$(\underline{C}_1, \underline{C}_2, \ldots, \underline{C}_n) = (\underline{Z}^T \underline{Z})^{-1} \underline{Z}^T \left[(\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_n) - \underline{Z}_2 (\underline{\Phi}_{21}^T, \underline{\Phi}_{22}^T, \ldots, \underline{\Phi}_{2n}^T) \right]$$

$$\underline{C} = (\underline{Z}^T \underline{Z})^{-1} \left\{ \underline{Z}^T \underline{W} - \underline{Z}^T \underline{Z}_2 \underline{\Phi}_2^T \right\} \qquad \text{(E.27)}$$

Also

$$\left.\begin{aligned} \underline{C}_1 &= \underline{\Phi}_{11}^T + \left(\underline{K}_R^{FB}\right)^T \underline{\Delta}_1^T \\ \underline{C}_2 &= \underline{\Phi}_{12}^T + \left(\underline{K}_R^{FB}\right)^T \underline{\Delta}_2^T \\ &\vdots \\ \underline{C}_n &= \underline{\Phi}_{1n}^T + \left(\underline{K}_R^{FB}\right)^T \underline{\Delta}_n^T \end{aligned}\right\} \quad (E.28)$$

$$\underline{C} = (\underline{C}_1, \underline{C}_2, \ldots, \underline{C}_n) = (\underline{\Phi}_{11}^T, \underline{\Phi}_{12}^T, \ldots, \underline{\Phi}_{1n}^T) + \left(\underline{K}_R^{FB}\right)^T (\underline{\Delta}_1^T, \underline{\Delta}_2^T, \ldots, \underline{\Delta}_n^T)$$

$$\underline{C} = \underline{\Phi}_1^T + \left(\underline{K}_R^{FB}\right)^T \underline{\Delta}^T \quad (E.29)$$

Using Equations (E.27) and (E.29),

$$\underline{\Phi}_1^T + \left(\underline{K}_R^{FB}\right)^T \underline{\Delta}^T = (\underline{Z}^T \underline{Z})^{-1} \left\{\underline{Z}^T \underline{W} - \underline{Z}^T \underline{Z}_2 \underline{\Phi}_2^T\right\}$$

which can be solved for $\underline{K}_R^{FB}$, if $(\Delta^T \Delta)^{-1}$ exists, as

$$\left(\underline{K}_R^{FB}\right)^T = \left\{(\underline{Z}^T \underline{Z})^{-1} (\underline{Z}^T \underline{W} - \underline{Z}^T \underline{Z}_2 \underline{\Phi}_2^T) - \underline{\Phi}_1^T\right\} \underline{\Delta} (\underline{\Delta}^T \underline{\Delta})^{-1} \quad (E.30)$$

The matrix $(\underline{\Delta}^T \underline{\Delta})^{-1}$ will usually exist. One case where it will <u>not</u> exist is if $\underline{\Delta}$ has a column of zeros, which is unlikely since then there is an input which does not affect the system.

In the calculation of Equation (E.30), the elements of $\underline{W}$ are evaluated using the high order optimal closed loop system, so that the rows of $\underline{W}$ are as

$$\underline{x}(j + 1) = (\underline{\Phi} + \underline{\Delta}\, \underline{K}^{FB})\, \underline{x}(j)$$

or

$$\underline{W} = (\underline{Z}, \underline{Z}_2) \begin{bmatrix} (\underline{\Phi}_1 + \underline{\Delta}\,\underline{K}_1^{FB})^T \\ (\underline{\Phi}_2 + \underline{\Delta}\,\underline{K}_2^{FB})^T \end{bmatrix}$$

$$\underline{W} = \underline{Z}\,(\underline{\Phi}_1 + \underline{\Delta}\,\underline{K}_1^{FB})^T + \underline{Z}_2\,(\underline{\Phi}_2 + \underline{\Delta}\,\underline{K}_2^{FB})^T \qquad \text{(E.31)}$$

Now Equation (E.30) can be written as

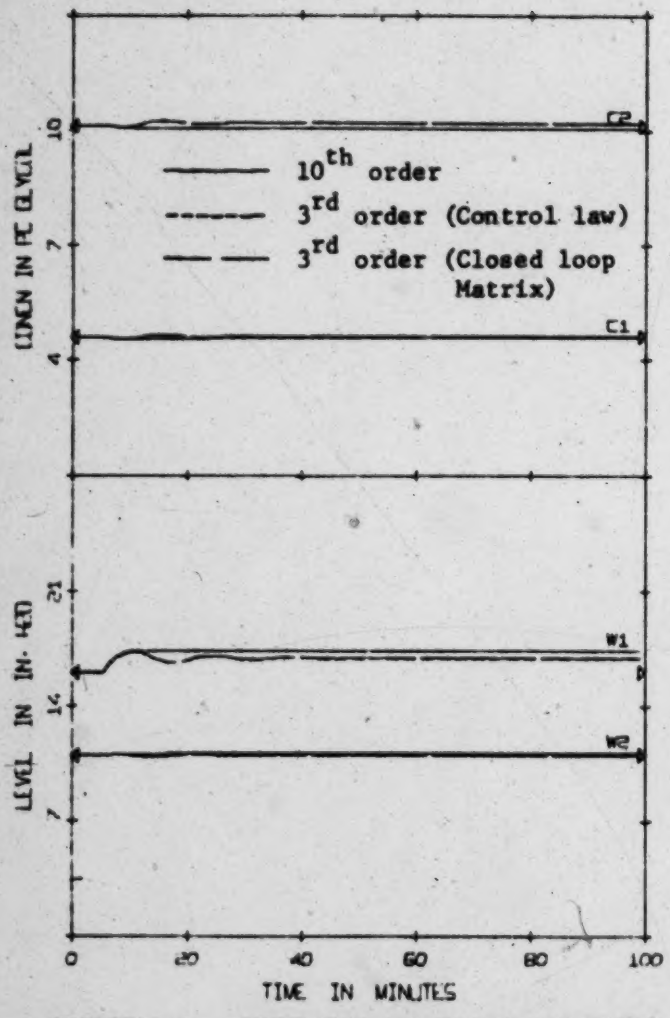
$$\left(\underline{K}_R^{FB}\right)^T = \left\{(\underline{Z}^T\,\underline{Z})^{-1}\left[\underline{Z}^T\,\underline{Z}\,(\underline{\Phi}_1 + \underline{\Delta}\,\underline{K}_1^{FB})^T + \underline{Z}^T\,\underline{Z}_2\,(\underline{\Phi}_2 + \underline{\Delta}\,\underline{K}_2^{FB})^T - \underline{Z}^T\,\underline{Z}_2\,\underline{\Phi}_2^T\right] - \underline{\Phi}_1^T\right\}\underline{\Delta}\,(\underline{\Delta}^T\,\underline{\Delta})^{-1}$$

$$= \left\{\underline{\Phi}_1^T + \left(\underline{K}_1^{FB}\right)^T\underline{\Delta}^T + (\underline{Z}^T\,\underline{Z})^{-1}\,\underline{Z}^T\,\underline{Z}_2\left(\underline{K}_2^{FB}\right)^T\underline{\Delta}^T - \underline{\Phi}_1^T\right\}\underline{\Delta}\,(\underline{\Delta}^T\,\underline{\Delta})^{-1}$$

$$= \left(\underline{K}_1^{FB}\right)^T + (\underline{Z}^T\,\underline{Z})^{-1}\,\underline{Z}^T\,\underline{Z}_2\,\left(\underline{K}_2^{FB}\right)^T \qquad \text{(E.32)}$$

which is identical to Equation (E.19) formed by the simpler approach of a least squares fit of Equation (E.1).

Figure E.1 shows a time domain response comparing the high order evaporator model optimally controlled with the high order model controlled by reduced order controllers calculated with each of these two approaches. The reduced order controllers give identical results.



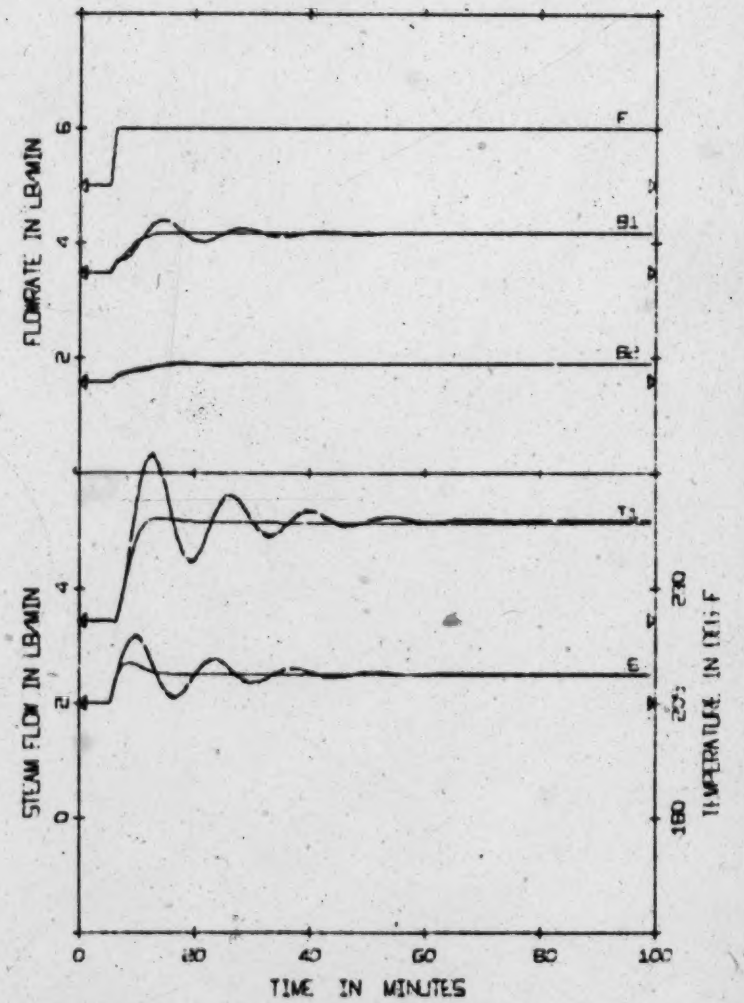


FIGURE E.1 SIMULATED COMPARISON OF THIRD ORDER CONTROL LAWS OBTAINED BY LEAST SQUARES REDUCTION (10(2018)/D, +20%F/FB/100PT, 3RED10(LS), 3RED10(LS)/3040A, 3034, 3050)

APPENDIX F

REDUCED ORDER MODELS

The reduced order models used in this thesis are shown in this Appendix. Table F.1 is an index. The models are presented in Tables F.2 to F.16.

TABLE F.1

INDEX TO REDUCED ORDER MODELS

Table No.	Reduction Method*	Model Order	C or D **	Run No. ***	Comments
F-2	MM	5	C	2020	
F-3	MM	4	C	2059A	Keeping eigenvalues 1,2,3,4
F-4	MM	4	C	2060A	Keeping eigenvalues 1,2,3,5
F-5	MM	3	C	2021	
F-6	MM	3	D	2050	
F-7	DM	3	D	2065	
F-8	FM	3	D	2094	
F-9	RLS1	3	D	2119	Start sequence at 19653
F-10	RLS1	3	D	2137	Start sequence at 9639
F-11	RLS1	3	D	2136	Start sequence at 1
F-12	RLS1	3	C	2141	Start sequence at 19653
F-13	DISC	3	D	2141	
F-14	RLS1	3	D	2113	Closed-loop model
F-15	RLS2	3	D	2108	Closed-loop model
F-16	TFLS	3	D	2144	Closed-loop model

* MM = Marshall's Modal Approach
DM = Davison's Modal Approach
FM = Fossard's Modal Approach
RLS1 = $\underline{\Phi}_R$ by random least squares, $\hat{\underline{\Delta}}_R$ for steady state.
RLS2 = $\underline{\Phi}_R$ and $\hat{\underline{\Delta}}_R$ by random least squares.
TFLS = $\underline{\Phi}_R$ by trajectory fitting least squares, $\hat{\underline{\Delta}}_R$ for steady state
DISC = Discrete version of model in Table F.12.

** C = Continuous-time model, D = discrete-time model

*** Simulated run in which the model was calculated.

TABLE F.2

FIFTH ORDER CONTINUOUS-TIME (MARSHALL'S) MODEL (2020)

$$\underline{A}_R = \begin{bmatrix} -0.5064E-06 & -0.3946E-03 & -0.4457E-01 & -0.9484E-07 & -0.6050E-06 \\ -0.1792E-04 & -0.7620E-01 & 0.4457E-01 & 0.9484E-07 & 0.6050E-06 \\ -0.1729E-06 & -0.1887E-02 & -0.2880E\ 00 & -0.3826E-06 & -0.2385E-05 \\ -0.2160E-07 & -0.7978E-03 & -0.9383E-01 & 0.1228E-06 & 0.4185E-06 \\ 0.2160E-07 & 0.3891E-01 & 0.9383E-01 & -0.2927E-04 & -0.3808E-01 \end{bmatrix}$$

$$\underline{B}_R = \begin{bmatrix} -0.9500E-02 & -0.7510E-01 & -0.2190E-08 \\ 0.9500E-02 & -0.1475E-02 & 0.2190E-08 \\ 0.1354E\ 00 & 0.5956E-02 & -0.8760E-08 \\ 0.8795E-02 & 0.8200E-01 & -0.3808E-01 \\ -0.8795E-02 & -0.4388E-01 & -0.1114E-07 \end{bmatrix}$$

$$\underline{D}_R = \begin{bmatrix} 0.1107E\ 00 & -0.2798E-04 & -0.3934E-02 \\ -0.3409E-01 & 0.7662E-01 & 0.3934E-02 \\ -0.1143E-01 & -0.2524E-03 & 0.5611E-01 \\ -0.7612E-03 & 0.2592E-04 & 0.3643E-02 \\ 0.7612E-03 & -0.2592E-04 & -0.3643E-02 \end{bmatrix}$$

TABLE F.3

FOURTH ORDER CONTINUOUS-TIME (MARSHALL'S) MODEL (2059A)
RETAINING THE FOURTH LARGEST EIGENVALUE

$$\underline{A}_R = \begin{bmatrix} -0.4531E-06 & -0.4134E-03 & -0.3609E-07 & -0.1576E-06 \\ 0.8195E-07 & -0.7672E-01 & -0.1015E-06 & -0.2173E-06 \\ 0.8613E-07 & -0.4555E-02 & 0.2416E-06 & 0.1324E-05 \\ -0.5233E-05 & -0.4260E\ 01 & -0.3507E-04 & -0.3812E-01 \end{bmatrix}$$

$$\underline{B}_R = \begin{bmatrix} -0.3031E-01 & -0.7602E-01 & 0.6932E-08 \\ 0.3589E-01 & 0.1590E-02 & 0.3487E-07 \\ -0.3328E-01 & 0.8016E-01 & -0.3808E-01 \\ 0.2044E\ 01 & 0.4610E-01 & -0.9107E-06 \end{bmatrix}$$

$$\underline{D}_R = \begin{bmatrix} 0.1124E\ 00 & -0.9437E-04 & -0.1255E-01 \\ -0.2812E-02 & -0.5700E-03 & 0.1486E-01 \\ 0.2885E-02 & -0.1082E-03 & -0.1378E-01 \\ -0.1771E\ 00 & 0.6519E-02 & 0.8466E\ 00 \end{bmatrix}$$

TABLE F.4

FOURTH ORDER CONTINUOUS-TIME (MARSHALL'S) MODEL (2060A)
RETAINING THE FIFTH LARGEST EIGENVALUE

$$\underline{A}_R = \begin{bmatrix} -0.4138E-06 & -0.4448E-01 & -0.9492E-07 & -0.6061E-06 \\ 0.2697E-06 & -0.2875E\ 00 & -0.3829E-06 & -0.2383E-05 \\ 0.1654E-06 & -0.9366E-01 & 0.1227E-06 & 0.4171E-06 \\ -0.9106E-05 & 0.8562E-01 & -0.2927E-04 & -0.3808E-01 \end{bmatrix}$$

$$\underline{B}_R = \begin{bmatrix} -0.9695E-02 & -0.7510E-01 & -0.2227E-08 \\ 0.1344E\ 00 & 0.5962E-02 & -0.8940E-08 \\ 0.8399E-02 & 0.8200E-01 & -0.3808E-01 \\ 0.1054E-01 & -0.4400E-01 & -0.7455E-08 \end{bmatrix}$$

$$\underline{D}_R = \begin{bmatrix} 0.1108E\ 00 & -0.4231E-03 & -0.4016E-02 \\ -0.1054E-01 & -0.2142E-02 & 0.5572E-01 \\ -0.3813E-03 & -0.7714E-03 & 0.3478E-02 \\ -0.1777E-01 & 0.3887E-01 & 0.4367E-02 \end{bmatrix}$$

TABLE F.5

THIRD ORDER CONTINUOUS-TIME (MARSHALL'S) MODEL (2021)

$\underline{A}_R$ =

-0.4534E-06	-0.3554E-07	-0.1532E-06
0.8207E-07	0.2477E-06	0.1369E-05
-0.9030E-05	-0.2938E-04	-0.3808E-01

$\underline{B}_R$ =

-0.3050E-01	-0.7603E-01	0.7019E-08
-0.3541E-01	0.8007E-01	-0.3808E-01
0.5060E-01	-0.4222E-01	-0.2524E-07

$\underline{D}_R$ =

0.1124E 00	-0.9173E-04	-0.1263E-01
0.3052E-02	-0.7374E-04	-0.1466E-01
-0.2091E-01	0.3823E-01	0.2095E-01

TABLE F.6

THIRD ORDER DISCRETE-TIME (MARSHALL'S) MODEL (2050)

$\underline{\Phi}_R$ =

0.9998E 00	-0.3795E-07	-0.2348E-06
0.8633E-07	0.9998E 00	0.1662E-05
-0.9408E-05	-0.3070E-04	0.9600E 00

$\underline{\Delta}_R$ =

-0.3250E-01	-0.8108E-01	0.8226E-08
-0.3770E-01	0.8539E-01	-0.4063E-01
0.5272E-01	-0.4413E-01	0.6023E-06

$\underline{\theta}_R$ =

0.1200E 00	-0.9780E-04	-0.1345E-01
0.3249E-02	-0.7866E-04	-0.1561E-01
-0.2177E-01	0.3979E-01	0.2183E-01

TABLE F.7

THIRD ORDER DISCRETE-TIME (DAVISON'S) MODEL (2065A)

$$\underline{\Phi}_R = \begin{bmatrix} 0.9999E\ 00 & -0.3791E-07 & -0.1602E-06 \\ 0.8753E-07 & 0.1000E\ 01 & 0.1431E-05 \\ -0.9439E-05 & -0.3071E-04 & 0.9601E\ 00 \end{bmatrix}$$

$$\underline{\Delta}_R = \begin{bmatrix} -0.3368E-01 & -0.8115E-01 & 0.2612E-07 \\ -0.3792E-01 & 0.8540E-01 & -0.4062E-01 \\ 0.7860E-01 & -0.4390E-01 & 0.6025E-06 \end{bmatrix}$$

$$\underline{\Theta}_R = \begin{bmatrix} 0.1200E\ 00 & -0.1459E-03 & -0.1391E-01 \\ 0.3255E-02 & -0.8020E-04 & -0.1564E-01 \\ -0.4117E-01 & 0.7942E-01 & 0.3251E-01 \end{bmatrix}$$

TABLE F.8

THIRD ORDER DISCRETE-TIME (FOSSARD'S) MODEL (2094)

$\underline{\underline{A}}_R$ =

0.9998E 00	-0.3795E-07	-0.2348E-06
0.8633E-07	0.9998E 00	0.1662E-05
-0.9408E-05	-0.3070E-04	0.9600E 00

$\underline{\underline{\Delta}}_R$ =

-0.3267E-01	-0.8110E-01	-0.5655E-09
-0.3767E-01	0.8539E-01	-0.4063E-01
0.7837E-01	-0.4388E-01	0.6037E-06

$\underline{\underline{\Theta}}_R$ =

0.1201E 00	-0.9744E-04	-0.1352E-01
0.3243E-02	-0.6977E-04	-0.1560E-01
-0.4103E-01	0.7915E-01	0.3242E-01

$\underline{\underline{E}}_R$ =

-0.6467E-02	0.1856E-03	0.3102E-01	0.7581E-01	0.2883E-02	-0.3164E-07
-0.1202E-01	-0.1916E-03	0.5879E-01	0.1430E 00	0.8932E-02	-0.7023E-07
0.4813E 00	-0.9833E 00	-0.2645E 00	-0.6410E 00	-0.6212E-02	-0.3328E-07

TABLE F.9

THIRD ORDER DISCRETE-TIME (RANDOM LEAST SQUARES) MODEL (2119), RANDOM SEQUENCE STARTING AT 19653

$\underline{\Phi}_R$ =

0.1000E 01	0.0000E 00	-0.2536E-03
0.0000E 00	0.1000E 01	-0.3933E-02
0.0000E 00	0.0000E 00	0.9661E 00

$\underline{\Delta}_R$ =

-0.3223E-01	-0.8132E-01	0.5584E-04
-0.3259E-01	0.8095E-01	-0.4060E-01
0.4451E-01	-0.3730E-01	0.9521E-05

$\underline{\Theta}_R$ =

0.1200E 00	0.7277E-04	-0.1333E-01
0.1100E-02	0.3811E-02	-0.1353E-01
-0.1840E-01	0.3360E-01	0.1843E-01

TABLE F.10

THIRD ORDER DISCRETE-TIME (RANDOM LEAST SQUARES) MODEL (2137), RANDOM SEQUENCE STARTING AT 9369

$\underline{\Phi}_R$ =

0.1000E 01	0.0000E 00	0.4113E-02
0.0000E 00	0.1000E 01	0.1017E-02
0.0000E 00	0.0000E 00	0.9597E 00

$\underline{\Delta}_R$ =

-0.3800E-01	-0.7650E-01	0.5572E-04
-0.3911E-01	0.8641E-01	-0.4060E-01
0.5292E-01	-0.4434E-01	0.9620E-05

$\underline{\Theta}_R$ =

0.1222E 00	-0.4270E-02	-0.1571E-01
0.3794E-02	-0.1111E-02	-0.1623E-01
-0.2187E-01	0.3995E-01	0.2191E-01

TABLE F.11

THIRD ORDER DISCRETE-TIME (RANDOM LEAST SQUARES) MODEL (2136), RANDOM SEQUENCE STARTING AT 1

$\underline{\Phi}_R$ =

0.1000E 01	0.0000E 00	-0.1348E-03
0.0000E 00	0.1000E 01	-0.1764E-02
0.0000E 00	0.0000E 00	0.9654E 00

$\underline{\Delta}_R$ =

-0.3239E-01	-0.8120E-01	0.5584E-04
-0.3545E-01	0.8334E-01	-0.4060E-01
0.4544E-01	-0.3807E-01	0.9532E-05

$\underline{\Theta}_R$ =

0.1200E 00	-0.4541E-04	-0.1339E-01
0.2280E-02	0.1654E-02	-0.1471E-01
-0.1878E-01	0.3430E-01	0.1881E-01

TABLE F.12

THIRD ORDER, CONTINUOUS-TIME (RANDOM LEAST SQUARES) MODEL (2141)

$\underline{A}_R$ =

0.0000E 00	0.0000E 00	0.1123E-01
0.0000E 00	0.0000E 00	0.2966E-02
0.0000E 00	0.0000E 00	-0.3869E-01

$\underline{B}_R$ =

-0.4544E-01	-0.6357E-01	0.2937E-06
-0.3936E-01	0.8335E-01	-0.3808E-01
0.5140E-01	-0.4292E-01	0.8379E-05

$\underline{D}_R$ =

0.1186E 00	-0.1137E-01	-0.1882E-01
0.4681E-02	-0.3050E-02	-0.1630E-01
-0.2126E-01	0.3884E-01	0.2127E-01

TABLE F.13

DISCRETE-TIME VERSION OF MODEL IN TABLE F.12

$\underline{\Phi}_R$ =

0.1000E 01	0.0000E 00	0.1174E-01
0.0000E 00	0.1000E 01	0.3099E-02
0.0000E 00	0.0000E 00	0.9595E 00

$\underline{\Delta}_R$ =

-0.4815E-01	-0.6807E-01	0.3662E-06
-0.4190E-01	0.8883E-01	-0.4062E-01
0.5371E-01	-0.4486E-01	0.8756E-05

$\underline{\theta}_R$ =

0.1264E 00	-0.1189E-01	-0.1994E-01
0.4958E-02	-0.3189E-02	-0.1735E-01
-0.2222E-01	0.4059E-01	0.2223E-01

TABLE F.14

THIRD ORDER DISCRETE-TIME CLOSED-LOOP (RANDOM LEAST SQUARES) MODEL (2113)

$\underline{\Phi}_R$ =

0.9904E 00	0.1276E-02	-0.5439E-03
0.1037E-01	0.9853E 00	0.4803E-03
-0.5908E-02	0.9245E-03	0.9947E 00

$\underline{\Delta}_R$ =

-0.2380E-02
-0.5682E-02
0.7190E-02

$\underline{\theta}_k$ =

0.1268E-01	0.5330E-03	-0.9857E-03
0.1227E-02	-0.4900E-03	-0.2354E-02
-0.2912E-02	0.5187E-02	0.2978E-02

TABLE F.15

THIRD ORDER, DISCRETE-TIME, CLOSED LOOP
(RANDOM LEAST SQUARES, $\underline{\phi}_R$, $\underline{\Delta}_R$ TOGETHER
MODEL (2108)

$\underline{\phi}_R$ =

0.9889E 00	-0.1394E-03	0.9264E-03
0.1112E-01	0.9851E 00	0.1374E-03
-0.6466E-02	0.8509E-03	0.9953E 00

$\underline{\theta}_R$ =

0.1541E-01
0.6797E-04
0.1565E-03

TABLE F.16

THIRD ORDER, DISCRETE-TIME, CLOSED LOOP
($\underline{\phi}_R$ BY TRAJECTORY FITTING, $\underline{\Delta}_R$ FOR STEADY STATE)
MODEL (2144)

$\underline{\phi}_R$ =

0.9404E 00	0.3393E-01	-0.3707E-02
0.1093E 00	0.8541E 00	-0.1276E-01
-0.1051E 00	0.1317E 00	0.1007E 01

$\underline{\Delta}_R$ =

0.2274E-02
-0.2279E-01
0.2579E-01

$\underline{\theta}_K$ =

0.4858E-01	0.3681E-02	0.9423E-03
-0.3614E-01	0.1267E-01	-0.9444E-02
0.3301E-01	-0.6986E-02	0.1068E-01